The Principle of Extremized Action

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Structure in a complex world

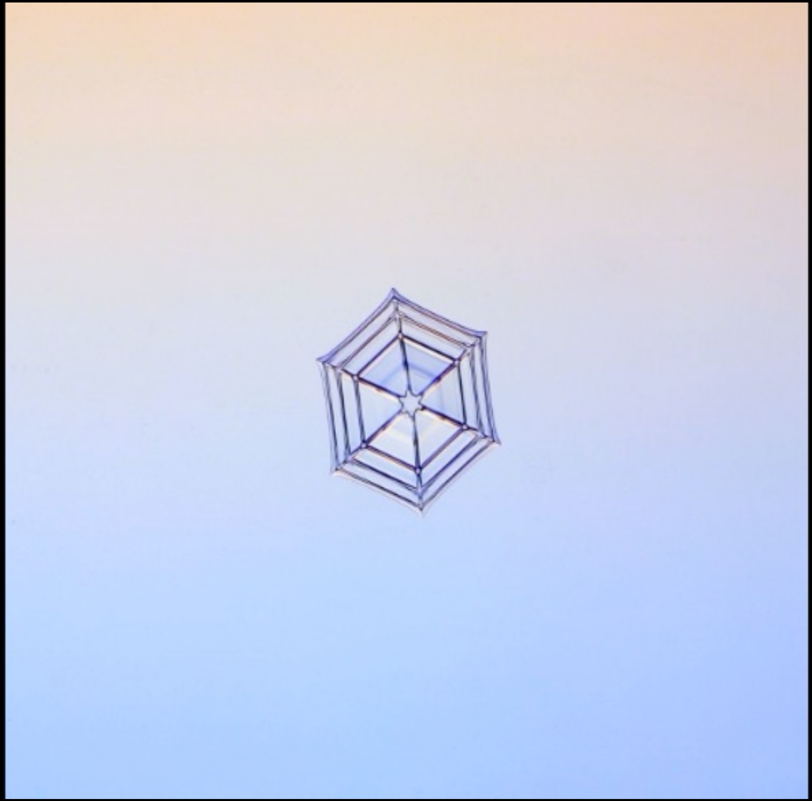
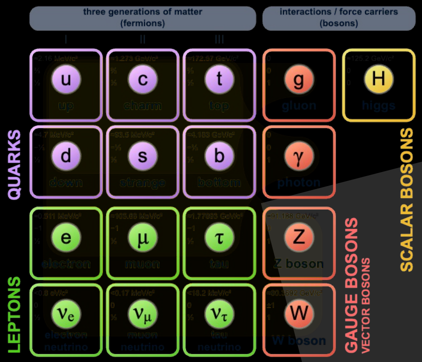
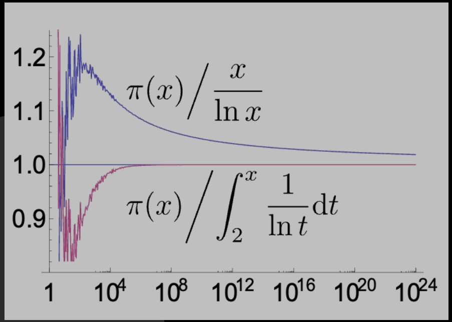


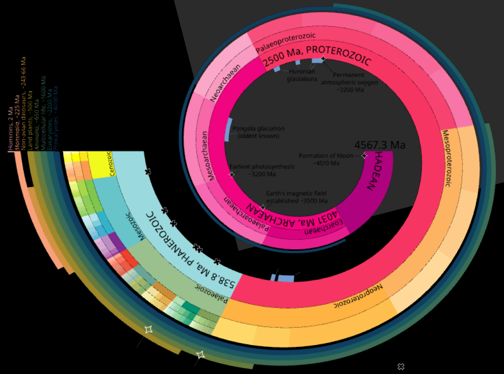
Image credit:
NASA, YouTube @harryshimmin
Kenneth G. Libbrecht (snowcrystals.com)

The search for structure





Science



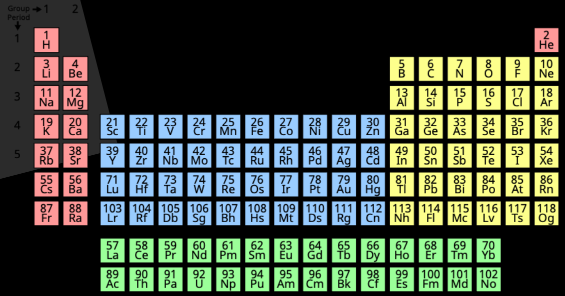


Image credit:
Wikipedia

Structure in Theoretical Physics

Guiding Principles

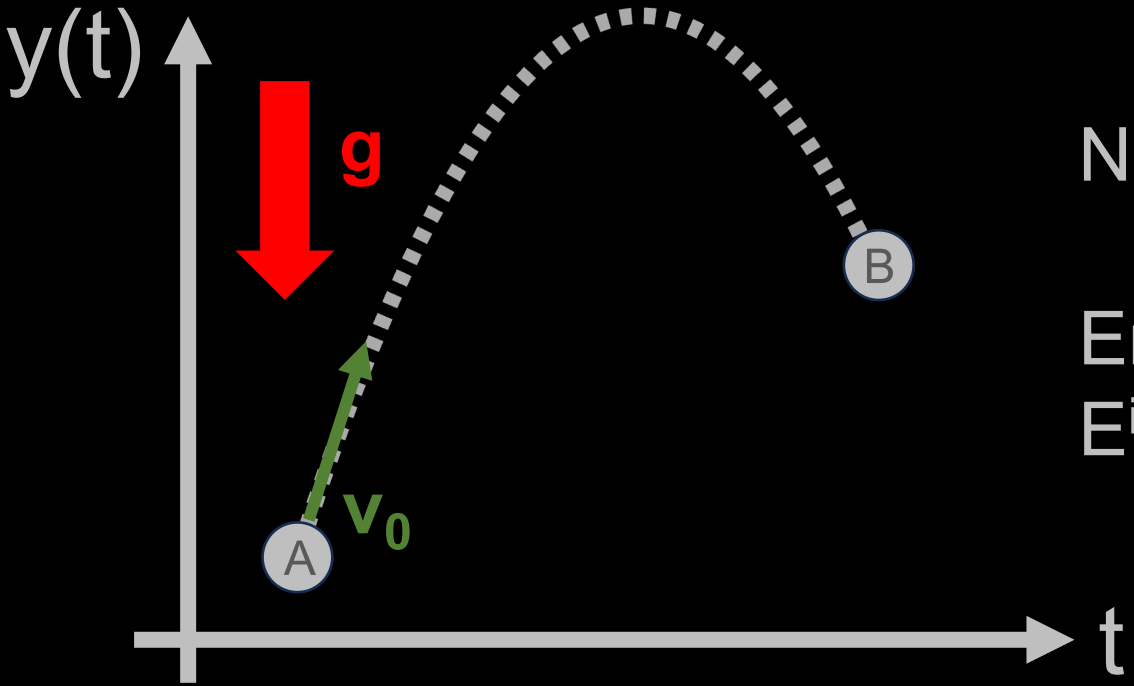
How → Why

Principle of extremized action

All of Classical Physics is Optimal

$$\delta S = 0$$

Describing motion - the how

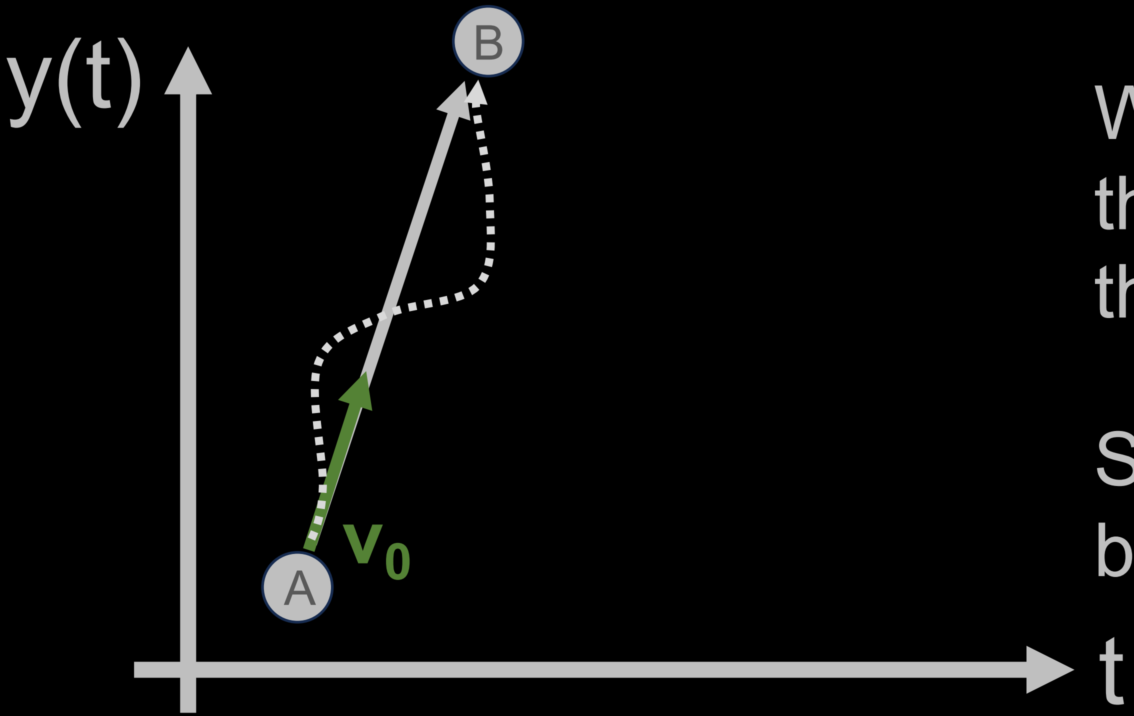


Newton's law: **F=ma**

Energy conservation

$E^i_{kin}+E^i_{pot}=E^f_{kin}+E^f_{pot}$

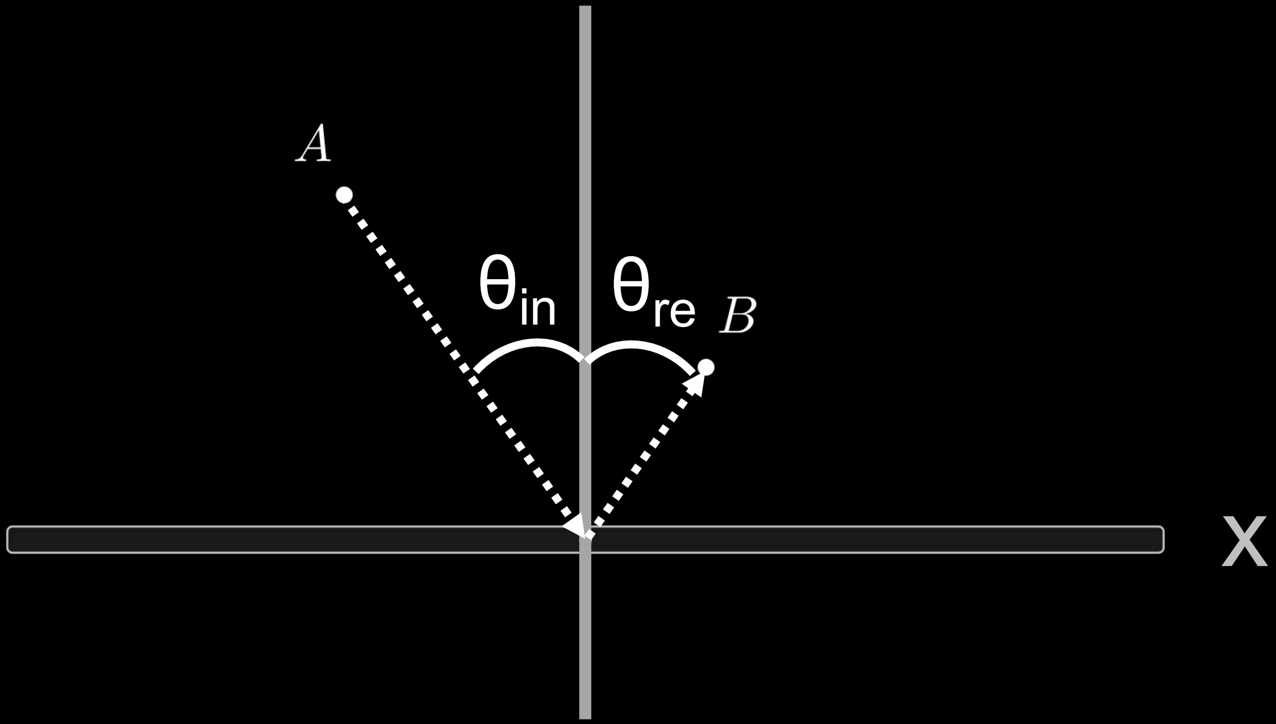
In the absence of forces



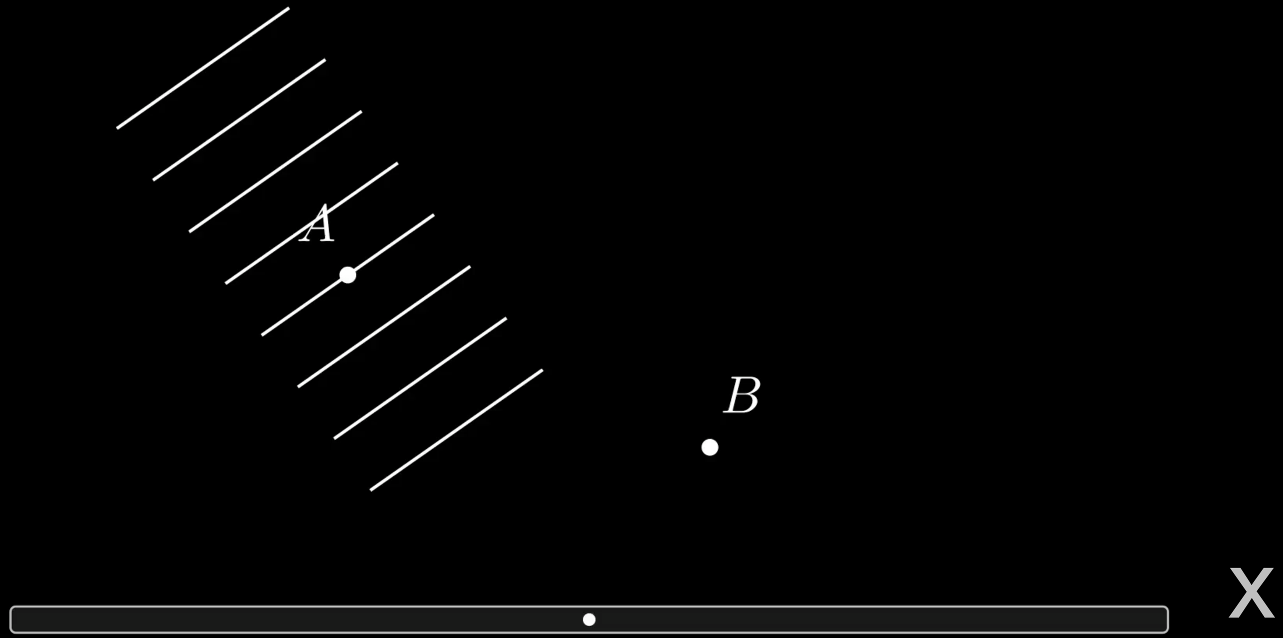
What **characterizes** the path taken by the object?

Shortest path between A and B

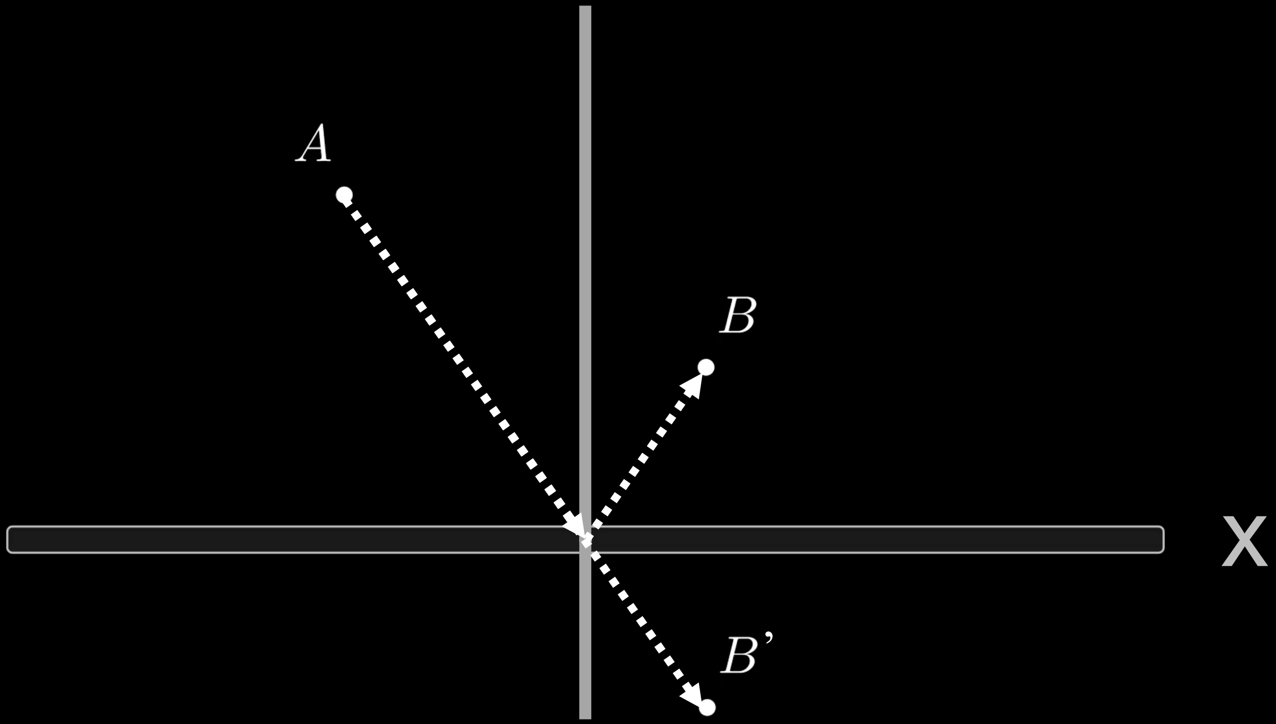
Reflection of a light ray – a recipe



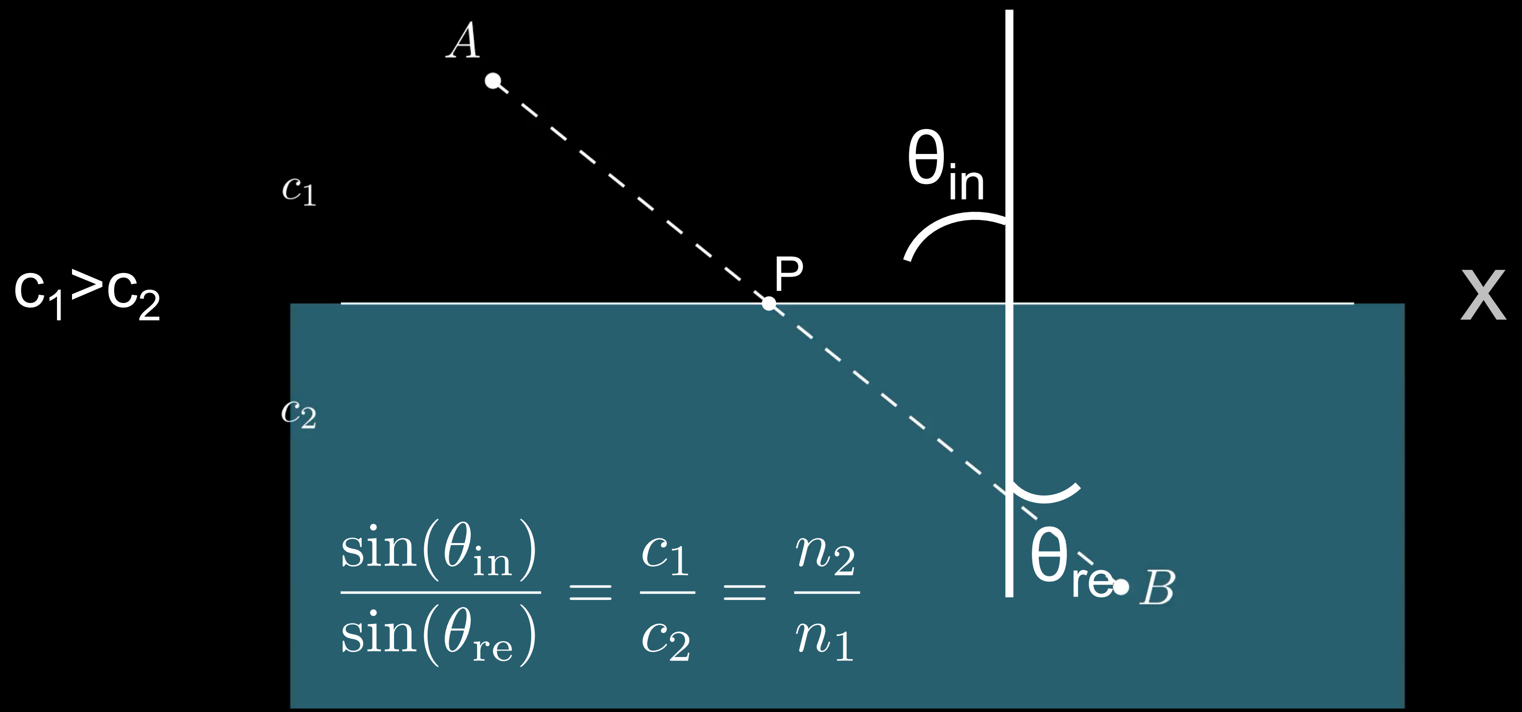
Reflection of a light ray – the how



Reflection of a light ray via guiding principle

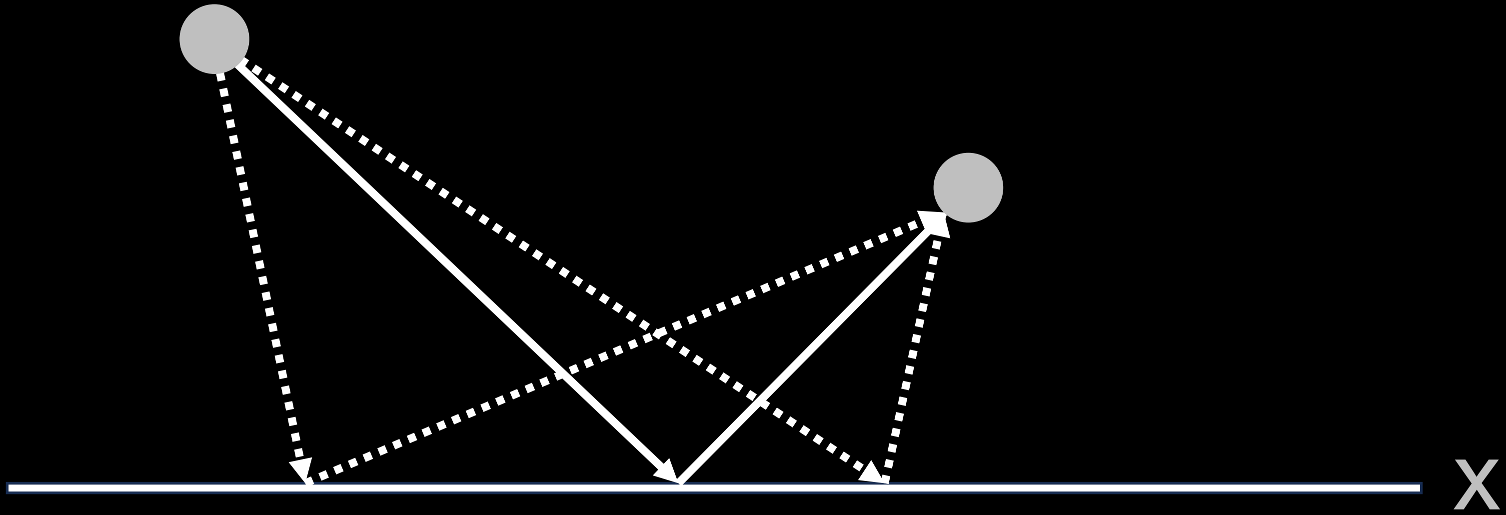


Refraction of a light ray



$$\frac{\sin(\theta_{\mathrm{in}})}{\sin(\theta_{\mathrm{re}})} = \frac{c_1}{c_2} = \frac{n_2}{n_1}$$

What characterizes point particle motion?



The further it travels – more action

The faster it covers a distance – more action

The heavier the object that travels – more action

Early attempts to define action

$$\int m\, v\, dx$$

Should be an intergral over the whole of motion

The longer it travels – more action
The faster it covers a distance – more action
The heavier the object that travels – more action

$$[\, m\ v\ dx\,] = \text{kg m}^2 / \text{s} = \text{kg m}^2/\text{s}^2 \cdot \text{s} = \text{J}\cdot\text{s}$$

The classical action – modern form

$$\int \frac{1}{2} m\, v\, dx$$

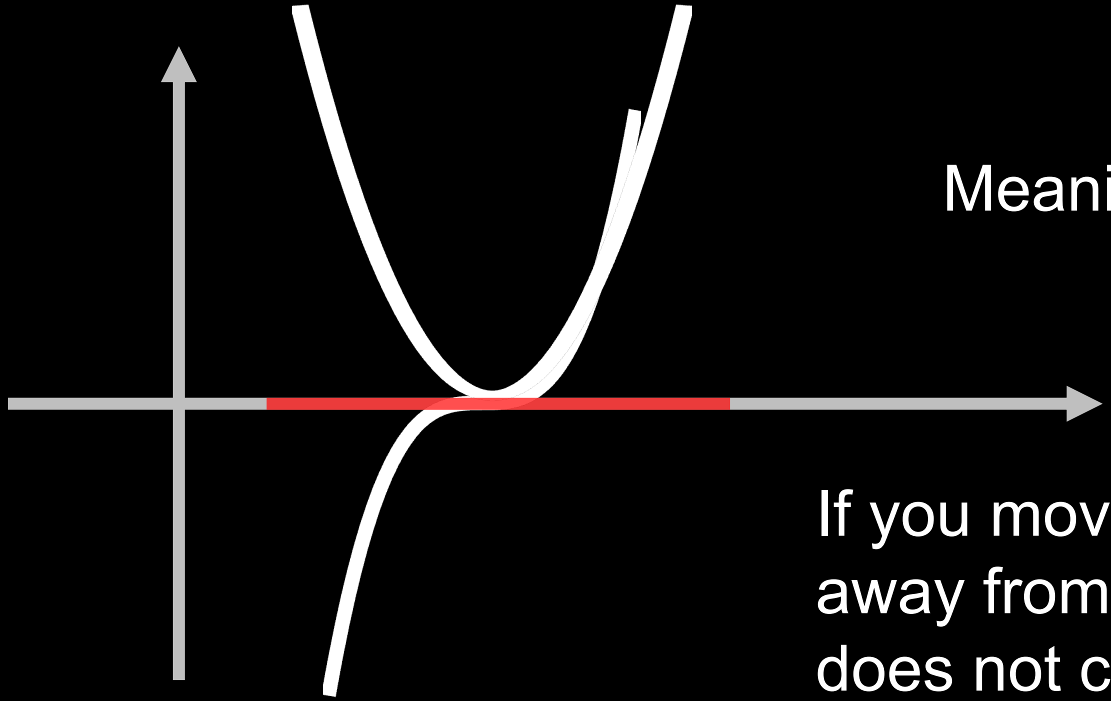
in absence of forces action is just kinetic energy

$$S = \int \left(E_{\rm kin} - E_{\rm pot} \right) dt$$

Classical action S = Score function for a path x(t)

How to get an optimal score?

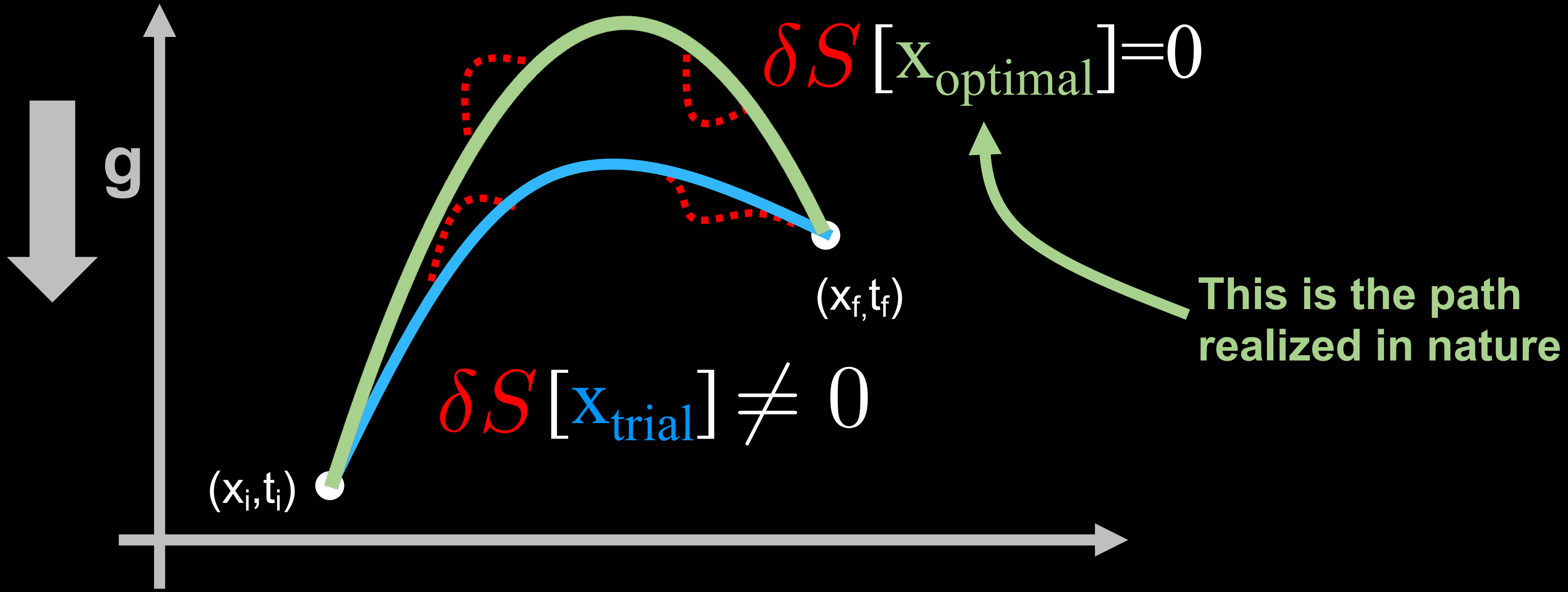
A familiar example: the extremum of a real function



Meaning of optimal?

If you move a tiny bit dx away from extremum x_0, f(x) does not change

Finding the optimal path



$$\delta S = S[\text{trial path}] - S[\text{trial path with tiny deformation}]$$

variational principle – extremized action

Rediscovering Newton’s law

$$\delta S = \int \left(\delta E_{\rm kin} - \delta E_{\rm pot}\right) dt$$

$$= \int \left(\left\{ - m\ddot{x} - \frac{dV}{dx}\right\}\delta x\right) dt \;\; = 0$$

$$-m\mathbf{a} + \mathbf{F} = 0$$

A guiding principle beyond mechanics

Electromagnetism

$$S = \int d^4x \, \frac{1}{2}\left(\varepsilon_0 \mathbf{E}^2 - \frac{1}{\mu_0}\mathbf{B}^2\right)$$

$$\delta S = 0$$

Maxwell's equation

$$\oiint_{\partial\Omega} \mathbf{E}\cdot \mathrm{d}\mathbf{S} = 4\pi \iiint_{\Omega} \rho \, \mathrm{d}V \qquad \oiint_{\partial\Omega} \mathbf{B}\cdot \mathrm{d}\mathbf{S} = 0$$

$$\oint_{\partial\Sigma} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Sigma} \mathbf{B}\cdot \mathrm{d}\mathbf{S} \qquad \oint_{\partial\Sigma} \mathbf{B}\cdot \mathrm{d}\boldsymbol{\ell} = \frac{1}{c}\left(4\pi \iint_{\Sigma} \mathbf{J}\cdot \mathrm{d}\mathbf{S} + \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Sigma} \mathbf{E}\cdot \mathrm{d}\mathbf{S}\right)$$

New insight from action principle

In General Physics I we encountered conserved quantities

energy E, momentum **p**, angular momentum L

Why not more, why not less?

Action principle relates conserved quantities and symmetries

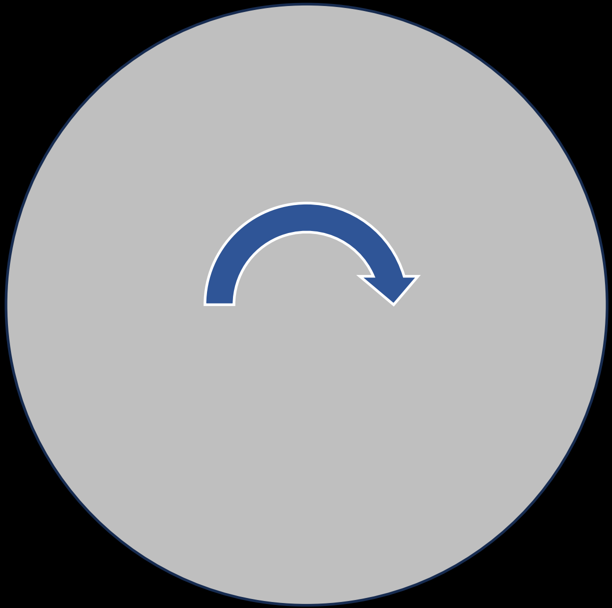
Towards Noether's theorem

Symmetries

(a way to transform a system leaving it unchanged)

$$\delta S_{\mathrm{T}} = 0$$

action invariant under symmetry



Noether's theorem

time translation symmetry (experiment today or next Wednesday)

Energy is conserved

Action unchanged under continuous symmetry

$$\delta S_{\mathrm{T}} = 0$$

$$\frac{dQ_T}{dt} = 0$$

rotation symmetry (experiment oriented in different ways)

L_x, L_y, L_z is conserved

spatial translation symmetry (experiment here or over there)

p_x, p_y, p_z is conserved

A glimpse into the quantum world

$$S$$

all of classical physics

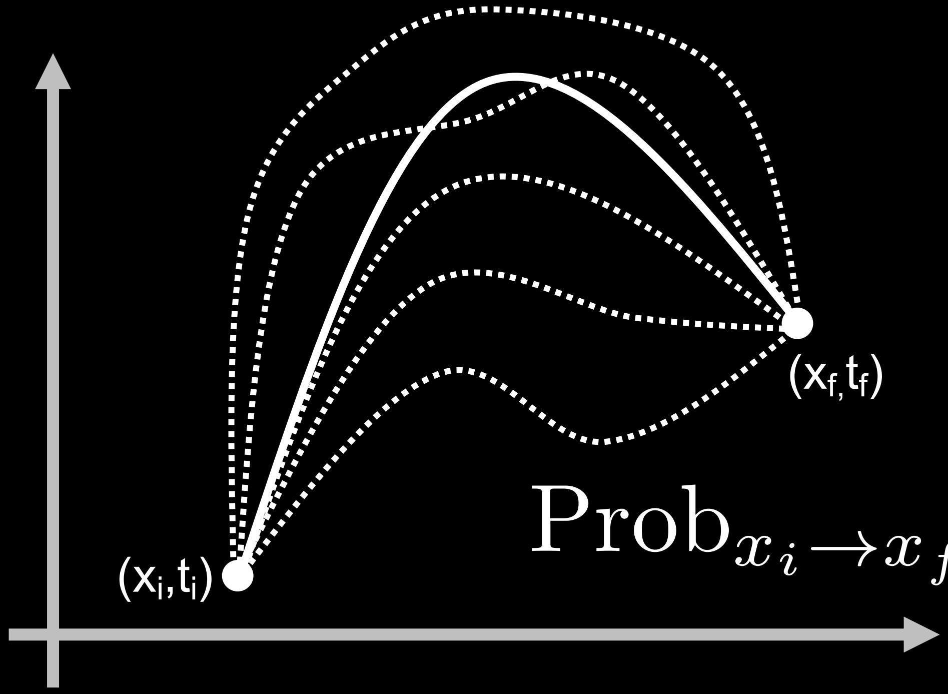
$$[S] = J \cdot s$$

$$\hbar$$

Planck's constant
Quantum physics

$$[\hbar] = J \cdot s$$

Classical action in the microscopic realm



$$\mathrm{Prob}_{x_i \to x_f} = \left| \sum_{\mathrm{paths}} e^{\frac{i}{\hbar} S[x,\dot{x}]} \right|^2$$

Summary

All classical motion can be understood as optimal w.r.t. the classical action (principle of extremized action)

$$S = \int (E_{\rm kin} - E_{\rm pot})dt \qquad \delta S = 0$$

The action reveals that continuous symmetries are intimately related to conserved quantities (Noether's theorem)

Even in the microscopic domain: classical action governs the quantum fluctuations of paths