Thermal Field Theory: (TFT)

Goal of this lecture: Explore basic concepts & techniques needed to understand and compute Thermodynamic properties of quantum fields (e.g. pressure)

- We will deploy TFT in its modern form: based on Feynman Path Int.
  -> Powerful, since applicable over vast energy ranges (cond. mat -> HEP)
  -> computationally efficient, since it treats $\mathbb{C}$-valued fields not operators.
  workhorse to study phase structure, charge transport etc.

- Thermal equilibrium: Idealized state, which is approached by any system with enough degrees of freedom at late enough time to arbitrary precision (fixed-point of quantum dynamics).

  In ideal TE: time translational invariance. At odds with time reversal invariance of microscopic evolution. Realized only approximately.
  Historic picture: small system in contact with heat bath, possible to exchange energy (canonical ensemble) or in addition particles (grand-canonical ensemble). [Ref: Le Bellac TFT; Kapusta & Gale; Laine Vuorinen; Yuhao Yang Diss.]

Characterized by partition function $Z_{GC}(T,\mu,V) = \mathrm{Tr}[\hat{\varrho}] = \mathrm{Tr}[e^{-\beta(\hat{H}-\mu\hat{N})}]$
$= \exp[-\Omega/T]$ Ω "thermodynamic potential" $\hat{H}$ Hamiltonian op.
$\beta = 1/T$ inverse temperature $\hat{N}$ number op. for conserved charge μ: "chemical potential" (how much E needed to introduce an additional charge).
$\hat{\varrho}$ density matrix $\hat{\varrho} = \sum_n p_n |n\rangle\langle n|$ "prob. of admissible realizations"
useful relations $\Omega = -PV$ (↑ pressure), alternatively via derivatives $P = T \frac{\partial \log Z}{\partial V}\Big|_T$
entropy: $S = -\frac{\partial \Omega}{\partial T}\Big|_{\mu,V}$ $N = -\frac{\partial \Omega}{\partial \mu}\Big|_{T,V}$ $E = \Omega + TS + \mu N$ (Legendre transform)
$= \Omega - T\frac{\partial \Omega}{\partial T} - \mu \frac{\partial \Omega}{\partial \mu}$
if $\mu = 0$ (gluons, photons) $\varepsilon = \frac{1}{V}\frac{\partial \log Z}{\partial \beta}\Big|_V$

Concrete examples:
- Early universe cosmology: equilibrium maintained by scatterings. Possible as long as int. rate $\Gamma = n\sigma v > H \sim T^2/m_{Pl}$ hubble rate of expansion.

chemical equilibrium: reactions that change part. number are fast enough so that $\mu_i + \mu_j = \mu_k + \mu_l$ for reaction $i+j \to k+l$. kinetic equilibrium: elastic reactions fast enough to maintain thermal momentum distribution.

"fast" refers to separation of timescales: microscopic $\sim \frac{1}{T}$ $H^{-1} \sim \frac{m_{pl}}{T^2}$ $T \sim 1\,\mathrm{MeV}$
because of freezing out of d.o.f. when microscopic timescales become large (weak int.)

- Heavy-ion collisions: Nuclear matter compressed & heated to $T \sim 10^{12}$ K.
Quarks and gluons liberated from confines of hadrons: quark-gluon plasma. Even though non-thermal: vital insight by idealized equilibrium physics. TE input, such as equation of state $P(T)$, $\epsilon(T)$ underpins successful phenomenological models of HIC (relativistic hydrodynamics). Other strategy measure & compute effect of hot environment on "impurities" such as pairs $\leftarrow Q\bar{Q}$ of heavy quarks and anti-quarks (quarkonium) {Ref: Rothkopf Phys Rep. 2020 general → Busza et al arXiv:1802.04801, Jaiswal, Roy arXiv:1605.08694} ← Hydro

- Cold quantum gases: Atomic fermi gases (^{6}Li, ^{40}K) at $<$ nK temperatures show pairing (BCS) and condensation phenomena (BEC) associated with different phases of the system. Encounter mixture of quantum and thermal phase transitions. Impurities in thermal medium play key role in thermometry (quantum Brownian motion).

Strategy in TFT: compute renormalized correlation functions of field dependent operators and relate to physical phenomena. [Note: Fruitful to engage with colleagues from experiment on what can be measured vs. computed]. Most relevant here one- and two-point functions $\langle\phi\rangle = \mathrm{Tr}[\hat{\rho}\hat{\phi}]$ $\langle\phi(t)\phi(t')\rangle = \mathrm{Tr}[\hat{\rho}\,\phi\phi]$

Path Integral for an initial value problem: $\hat{\phi}|\phi\rangle = \phi(x)|\phi\rangle$ $\pi = \frac{\partial \mathcal{L}}{\partial \dot\phi}$
$\hat{\pi}|\pi\rangle = \pi(x)|\pi\rangle$ $[\hat{\phi}, \hat{\pi}] = -i\delta^{(3)}(\vec{x}-\vec{y})$

$$\mathrm{Tr}[\hat{\rho}(t)\,\hat{O}] = \int d\phi\, \langle\phi|U^\dagger(t,0)\,\hat{\rho}_0\, U(t,0)\,\hat{O}|\phi\rangle = \int d\phi \int d\phi' \int d\phi''\, \langle\phi|\hat{\rho}_0|\phi'\rangle\langle\phi'|U(t,0)|\phi''\rangle\langle\phi''|\hat{O}\,U^\dagger(t,0)|\phi\rangle$$

evolution of density matrix introduces two time evolution operators. To obtain PI chop up $U(t,0)$ into infinitesimal pieces and insert complete set $|\phi\rangle, |\pi\rangle$

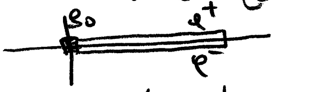
$$\langle\phi_j|e^{-iHt}|\phi_i\rangle = \lim_{N\to\infty} \int \prod_{k=1}^{N} d\pi_k\, d\phi_k/2\pi\ \langle\phi_j|\pi_N\rangle\langle\pi_N|e^{-iH\Delta t}|\varphi_N\rangle\langle\varphi_N|\pi_{N-1}\rangle\langle\pi_{N-1}|e^{-iH\Delta t}\cdots$$
$$\langle\phi_2|\pi_1\rangle\langle\pi_1|e^{-iH\Delta t}|\phi_1\rangle\underbrace{\langle\phi_1|\varphi_i\rangle}_{\delta(\phi_1-\phi_i)}$$

$$\langle\pi_i|e^{-iH\Delta t}|\phi_i\rangle \approx \langle\pi_i|(1 - i\hat{H}\Delta t)|\phi_i\rangle = \underbrace{\langle\pi_i|\phi_i\rangle}_{\exp[-i\int\pi_i\phi_i]}(1 - iH(\phi_i,\pi_i)\Delta t)$$

$$= \lim_{N\to\infty} \int \prod_k d\pi_k\, d\phi_k/2\pi\ \delta(\varphi_1-\varphi_i)\, \exp\Big[-i\Delta t \sum_j \int d^3x\, \big[H(\phi_j,\pi_j) - \pi_j(\phi_{j+1}-\phi_j)/\Delta t\big]\Big]$$

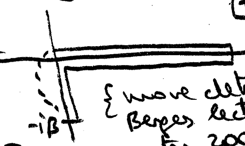
{for more details: Orland & Negele 1998}

$\mathrm{Tr}[\hat{\varrho}\,\hat{\varphi}(t)] = \underbrace{\int d\varphi \int d\varphi' \langle\varphi|\varrho_0|\varphi'\rangle}_{\text{initial cond}} \underbrace{\int D\pi \int D\varphi\, \exp\left[i\int d^4x\,(\pi_+\dot{\phi}_+ - H(\phi_+,\pi_+)) - i\int d^4x\,(\pi_-\dot{\phi}_- - H(\phi_-,\pi_-))\right]}_{\text{quantum dyn.}}$



Therm. equilibrium special: $e^{-\beta\hat{H}} = U(-i\beta,0)$ reinterpreted as evolution operator in imaginary time $\tau=-it$.* If we can integrate π's

$\mathrm{Tr}[\hat{\varrho}\,\hat{\varphi}(t)] = \int D\varphi_E \int D\varphi_\pm\, e^{iS_M[\varphi_+] - iS_M[\varphi_-] - S_E[\varphi_E]}$ underlying time contour



* because of trace identity $\varphi_E(\tau=\beta) = \varphi(0)$ compact & periodic

⇒ Schwinger-Keldysh time contour (closed time path) {more details Berges lecture notes 2004}

You may choose any contour as long as inside strip $\tau\in[0,\beta]$ hits downward and ends at $-i\beta$. Since expectation values are time translation invariant ($[\hat{\varrho},\hat{H}]=0$) starting point of contour arbitrary.

Relevant quantities: At T=0 scattering matrix elements for asympt. states. At T≠0 interactions with medium anywhere, concept of asymptotic state void. Instead Wightman functions: $D^>(t,t_0) = \langle\hat{\varphi}(t)\hat{\varphi}(t_0)\rangle$ $D^<(t,t_0) = \langle\hat{\varphi}(t_0)\hat{\varphi}(t)\rangle$ describe correlations or retarded/advanced correlator

$D^R(t,t_0) = \Theta(t-t_0)\langle[\hat{\varphi}(t),\hat{\varphi}(t_0)]\rangle$ $D^A(t,t_0) = \Theta(t_0-t)\langle[\hat{\varphi}(t),\varphi(t_0)]\rangle$ which encode causal relationship.

In TE all of these are related by the Kubo-Martin-Schwinger (KMS) relation, a representation of the fluctuation-dissipation theorem.

$D^>(t_1,t_0) = \frac{1}{Z}\sum_n \langle n|e^{-\beta\hat{H}}\varphi(t_1)\varphi(t_0)|n\rangle = \frac{1}{Z}\sum_{n,m} e^{-\beta E_n} e^{-iE_n(t_1-t_0)} e^{iE_m(t_0-t_1)} |\langle n|\varphi(0)|m\rangle|^2$

As long as $-\beta < \mathrm{Im}(t) < 0$ sum is absolutely convergent: analytic in that strip.

⇒ $D^>(t=t_1-t_0) = D^<(t+i\beta) \Leftrightarrow D^>(\omega) = e^{\beta\omega}D^<(\omega)$

All correlators can be expressed in terms of one quantity: spectral function

$\varrho(t_1,t_0) = \langle[\varphi(t_1),\varphi(t_0)]\rangle = D^R(t_1,t_0) - D^A(t_1,t_0) = D^>(t_1,t_0) - D^<(t_1,t_0) = -\mathrm{Im}[D^R(\omega)]$

⇒ $D^>(\omega) = (1+n_B(\omega))\varrho(\omega)$ $D^<(\omega) = n_B(\omega)\varrho(\omega)$ $D^R(t_1,t_0) = \Theta(t_1-t_0)\varrho(t_1,t_0)$ *

Since analytic functions: may evaluate in Euclidean time to simplify computations. $n_B(\omega) \overset{*}{=} \frac{1}{e^{\beta\omega}-1}$

$D(\tau) = \langle\hat{\varphi}(\tau_1)\hat{\varphi}(\tau_0)\rangle = D^>(-i\tau = -i(\tau_1-\tau_0))$ $0<\tau<\beta$ Fourier transform on finite interval leads to discrete Matsubara frequencies $\omega_n = 2\pi n/\beta = 2\pi n T$.

Useful relation expressed via Lehmann representation

$D_M(\omega_n) = \int d\omega \frac{1}{\omega - i\omega_n}\varrho(\omega)$ $D_E(\tau) = \frac{1}{\pi}\int d\omega \frac{e^{-\tau\omega}}{1-e^{-\beta\omega}}\varrho(\omega)$

spectral function acts as bridge between Euclidean and real-time domain.

$D^{R/A}(p_0) = \pm i\int \frac{d\omega}{\pi}\frac{1}{p_0-\omega\pm i\varepsilon}\varrho(\omega) = \mp i D_M(i(p_0 \pm i\varepsilon))$

For Fermions: $S^> = \langle \psi(t_1)\bar\psi(t_0)\rangle$ $\quad S^< = -\langle\bar\psi(t_0)\psi(t_1)\rangle$

$S^>(t) = -e^{-\beta\mu} S^<(t+i\beta)$ $\quad S^>(\omega) = -e^{\beta(\omega-\mu)} S^<(\omega)$ $\quad n_F(\omega)=\frac{1}{1+e^{\beta\omega}}$

$S^<(\omega) = -n_F(\omega-\mu)\,\rho_F(\omega)$ $\quad \rho_F(t_1,t_0)=\langle\{\psi(t_1),\bar\psi(t_0)\}\rangle$

$S_E(\tau)=\int d\omega\, \frac{e^{-\tau\omega}}{1+e^{-\beta\omega}}\rho_F(\omega)$

<u>1st challenge</u>: Compute the <u>pressure</u> for a charged <u>scalar</u> <u>field</u> theory exploiting the <u>Euclidean</u> formulation. $Z = \mathrm{Tr}[e^{-\beta(\hat H-\mu\hat N)}] = \int D\phi_E D\pi_E \exp[i\pi\frac{\partial\phi}{\partial\tau} - H + \mu N]$

$$\mathcal L = \partial_\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 \Rightarrow j^\mu = i(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*) \quad Q=\int d^3x\, j^0$$

conserved charge implies existence of chemical potential μ. Most convenient

$\phi = \varphi_1 + i\varphi_2$ $\quad \hat{\mathcal H} = \frac12[\hat\pi_1^2+\hat\pi_2^2+(\nabla\hat\varphi_1)^2+(\nabla\hat\varphi_2)^2+m^2\hat\varphi_1^2+m^2\hat\varphi_2^2] + \frac14\{\lambda(\hat\varphi_1^2+\hat\varphi_2^2)^2\}$

$$\hat Q = \int d^3x\,(\hat\varphi_2\hat\pi_1 - \hat\varphi_1\hat\pi_2)$$

$$Z = \int D\pi_1 D\pi_2 \int D\varphi_1 D\varphi_2 \exp\left[\int_0^\beta d\tau\int d^3x\left\{i\pi_1\frac{\partial\varphi_1}{\partial\tau} + i\pi_2\frac{\partial\varphi_2}{\partial\tau} - \mathcal H(\pi_1,\pi_2,\varphi_1,\varphi_2) + \mu(\varphi_2\pi_1-\varphi_1\pi_2)\right\}\right]$$

complete the square and carry out Gaussian integrals in π

$$= \mathcal N'\int D\varphi_1 D\varphi_2 \exp\left[\int_0^\beta d\tau\int d^3x\left\{-\frac12\left(\frac{\partial\varphi_1}{\partial\tau}-i\mu\varphi_2\right)^2 - \frac12\left(\frac{\partial\varphi_2}{\partial\tau}+i\mu\varphi_1\right)^2 - \frac12\left((\nabla\varphi_1)^2+(\nabla\varphi_2)^2\right)\right.\right.$$
$$\left.\left. -\frac12 m^2(\varphi_1^2+\varphi_2^2) - \frac{\lambda}{4}(\varphi_1^2+\varphi_2^2)^2\right\}\right]$$

Check free field limit first $\lambda=0$. Anticipating that Bosons prefer to populate ground state use following ansatz: (placed in a <u>large but finite box</u>)

$$\varphi_1 = \sqrt2\,\xi\cos\theta + \sqrt{\tfrac{\beta}{V}}\sum_{n=1}^\infty\sum_{\vec p} e^{i(\vec p\vec x+\omega_n\tau)}\varphi_{1,n}(\vec p) \quad \text{(mode expansion)}$$

$$\varphi_2 = \underbrace{\sqrt2\,\xi\sin\theta}_{\text{separate zero mode}} + \sqrt{\tfrac{\beta}{V}}\sum_n\sum_{\vec p} e^{i(\vec p\vec x+\omega_n\tau)}\varphi_{2,n}(\vec p) \quad \text{Rewrite PI in terms of } \varphi(\vec p)$$

at $T=0$ diagonalizes exponent but at $T>0$ we have: $\quad$ w/ $\omega^2=\vec p^2+m^2$

$$Z = \mathcal N'\int D\varphi_1(p)\int D\varphi_2(p)\exp\left[\beta V(\mu-m^2)\xi^2 - \frac12\sum_n\sum_{\vec p}(\varphi_{1,n}(-p),\varphi_{2,n}(-p))\,D\begin{pmatrix}\varphi_{1,n}(p)\\ \varphi_{2,n}(p)\end{pmatrix}\right]$$

$$D = \beta^2\begin{pmatrix}\omega_n^2+\omega^2-\mu^2 & -2\mu\omega_n\\ 2\mu\omega_n & \omega_n^2+\omega^2-\mu^2\end{pmatrix}$$ Gaussian int: $\ln Z = c+\beta V(\mu^2-m^2)\xi^2+\ln(\det D^{-1/2})$

Factoring the determinant we get

$$\ln Z = \beta V(\mu^2-m^2)\xi^2 - \frac12\sum_n\sum_{\vec p}\left\{\ln[\beta^2(\omega_n^2+(\omega-\mu)^2)] + \ln[\beta^2(\omega_n^2+(\omega+\mu)^2)]\right\}$$

How to compute sum $\sum_n$? First trick "logarithm acrobatics"

$$\ln[(2\pi n)^2+\beta^2\omega^2] = \int_1^{\beta^2\omega^2}\frac{d\theta^2}{\theta^2+(2\pi n)^2} + \ln[1+(2\pi n)^2] \quad \& \quad \sum_{n=-\infty}^{\infty}\frac{1}{n^2+(\theta/2\pi)^2} = \frac{2\pi^2}{\theta}\left(1+\frac{2}{e^\theta-1}\right)$$

using some logarithm acrobatics and following summation formula

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-x)(n-y)} = \frac{\pi(\cot \pi x - \cot \pi y)}{y-x}$$ you will obtain

$$\ln Z = 2V \int \frac{d^3p}{(2\pi)^3} \Big[\beta\omega + \underbrace{\ln(1+e^{-\beta(\omega-\mu)}) + \ln(1+e^{-\beta(\omega+\mu)})}_{\text{particle \& antiparticle contribution}} \Big]$$

2: automatic due to spinor structure; $\beta\omega$: zero point energy

Onwards to the pressure in interacting theory: Perturbation theory

$$Z = N' \int D\varphi\, e^{-S_0} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} S_I^\ell \Rightarrow \ln Z = \underbrace{\ln\left(N'\int D\varphi\, e^{-S_0}\right)}_{\ln Z_0} + \underbrace{\ln\left(1 + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \frac{\int D\varphi\, e^{-S_0} S_I^\ell}{\int D\varphi\, e^{-S_0}}\right)}_{\ln Z_I}$$

Take a look at $\ln Z_I$ in $\lambda\varphi^4$ theory: first term $\quad (\log(1+x) \approx x - x^2/2 + \dots)$

$$\ln Z_1 = -\lambda \int d\tau \int d^3x \int D\varphi\, e^{-S_0} \varphi^4(\vec{x},\tau) \Big/ \int D\varphi\, e^{-S_0}$$ rewrite using Fourier transform

$$= -\lambda \int d\tau \int d^3x \sum_{n_1 \dots n_4} \sum_{\vec{p}_1 \dots \vec{p}_4} \frac{\beta^2}{V^2} \exp\left(i(\vec{p}_1 + \dots + \vec{p}_4)\vec{x}\right) \exp\left(i(\omega_{n_1} + \dots + \omega_{n_4})\tau\right) \frac{I_1}{I_2}$$

$$I_1 = \underbrace{\prod_{\ell} \prod_{\vec{q}} \int d\varphi_\ell(\vec{q}) \exp\left[-\tfrac{1}{2}\beta^2(\omega_\ell^2 + \vec{q}^{\,2} + m^2)\varphi_\ell(\vec{q})\varphi_{-\ell}(-\vec{q})\right]}_{I_2} \varphi_{n_1}(\vec{p}_1)\varphi_{n_2}(\vec{p}_2)\varphi_{n_3}(\vec{p}_3)\varphi_{n_4}(\vec{p}_4)$$

~~$I_2 = \prod_\ell \prod_{\vec{q}}$~~

Integration over $\tau, \vec{x}$ yields δ-functions which require two of φ's to combine to

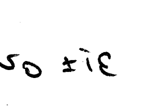
$$\ln Z_1 = \overset{\delta\text{-funct poss. to combine } p_i\text{'s and } \omega_i\text{'s}}{3} \frac{1}{Z_0} (-\lambda) \beta \overset{\delta\text{-func norm.}}{V} \prod_{\ell,\vec{q}} \int d\varphi_\ell(\vec{q}) \frac{\beta^2}{V^2} \sum_{n,m} \sum_{p,p'} |\varphi_n(p)|^2 |\varphi_m(p')|^2 \exp\left[-\tfrac{1}{2}\beta^2(\omega_\ell^2 + \vec{q}^{\,2} + m^2)|\varphi_\ell(\vec{q})|^2\right]$$

remember that $\int dx\, x^4 e^{-ax^2} \big/ \int dx\, e^{-ax^2} = \frac{3}{4}\frac{1}{a^2}$ so we get

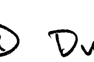
$$= -3\lambda\beta V \left(\frac{V}{\beta} \sum_{\ell,\vec{q}} \frac{1}{\omega_\ell^2 + \vec{q}^{\,2} + m^2}\right)^2 = -3\lambda\beta V \Big[\frac{1}{\beta} \sum_\ell \int \frac{d^3q}{(2\pi)^3} \underbrace{\frac{1}{\omega_\ell^2 + \vec{q}^{\,2} + m^2}}_{D_0(\omega_n, \vec{q})}\Big]^2$$

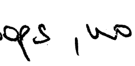
similar to vacuum PT $D_0(\omega_n, \vec{q})$ propagator of free theory basic ingredient. No $\pm i\epsilon$ since rotated to imag. Matsubara freq. Diagrammatic representation via Feynman diagrams. Remember from $T=0$: $1/Z_0$ denom. cancels disconn. diags.

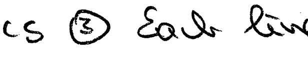
<u>Feynman rules</u>: ① Draw all conn. diagrams from contracting φ fields (here only closed loops, no in- out states, no LSZ!). ② Determine combinatorics ③ Each line gets $T \sum_n \int d^3p/(2\pi)^3 D_0(\omega_n, \vec{p})$ ④ $-\lambda$ for each vertex

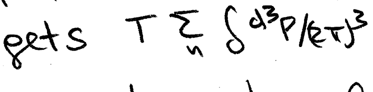


⑤ energy momentum conservation at each vertex, $(2\pi)^3\delta(\vec{p}_{in} - \vec{p}_{out})\beta\delta_{\omega_{in},\omega_{out}}$ lead to overall $\beta(2\pi)^3\delta(0) = \beta V$ as overall factor.

$\ln Z_1 = -3 \times (\beta V) \times$ . Next order: $\ln Z_2 = -\frac{1}{2}\underbrace{\left(\frac{\int D\varphi\, e^{-S_0} S_I}{\int D\varphi\, e^{-S_0}}\right)^2}_{(\ln Z_1)^2} + \frac{1}{2}\frac{\int D\varphi\, e^{-S_0} S_I^2}{\int D\varphi\, e^{-S_0}}$

second term from : three topologically distinct diagrams

$+\frac{1}{2}$  possible contr.

 cancels the first term.

guessing the combinatorics tricky, use Wicks theorem: contract distinct pairs

$$\langle \varphi(x)\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)\varphi(y)\rangle_c = 4\langle\varphi(x)\varphi(y)\rangle_0\langle\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)\rangle_c$$
$$+3\langle\varphi(x)\varphi(x)\rangle_0\langle\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)\varphi(y)\rangle_c$$

first term removes combinations where fields contracted at same point.

$$= 4\times 3\ \langle\varphi(x)\varphi(y)\rangle_0\langle\varphi(x)\varphi(y)\rangle_0\langle\varphi(x)\varphi(x)\varphi(y)\varphi(y)\rangle_c$$
$$+4\times 2\ \langle\varphi(x)\varphi(y)\rangle_0\langle\varphi(x)\varphi(x)\rangle_0\langle\varphi(x)\varphi(y)\varphi(y)\varphi(y)\rangle_c$$
$$+3\times 4\ \langle\varphi(x)\varphi(x)\rangle\langle\varphi(x)\varphi(y)\rangle_0\langle\varphi(x)\varphi(y)\varphi(y)\varphi(y)\rangle_c$$

$$= 4\times3\times2\ \langle\varphi(x)\varphi(y)\rangle_0\langle\varphi(x)\varphi(y)\rangle_0\langle\varphi(x)\varphi(y)\rangle_0\langle\varphi(x)\varphi(y)\rangle_0 + 4\times 3\ \langle\varphi(x)\varphi(y)\rangle\langle\varphi(x)\varphi(y)\rangle\langle\varphi(x)\varphi(x)\rangle\langle\varphi(y)\varphi(y)\rangle$$
$$+4\times2\times3\ \langle\varphi(x)\varphi(y)\rangle\langle\varphi(x)\varphi(x)\rangle\langle\varphi(x)\varphi(y)\rangle_0\langle\varphi(y)\varphi(y)\rangle_0$$
$$+3\times4\times3\ \langle\varphi(x)\varphi(x)\rangle_0\langle\varphi(x)\varphi(y)\rangle_0\langle\varphi(x)\varphi(y)\rangle_0\langle\varphi(y)\varphi(y)\rangle_0 \Rightarrow 24\ \ominus + 72\ \bigcirc\!\bigcirc\!\bigcirc$$

$$\Rightarrow \ln Z_2 = 36\,(\lambda^2)\,(\beta V)^2\ \bigcirc\!\bigcirc\!\bigcirc + 12\,(\lambda^2)\,(\beta V)^2\ \ominus$$

Now let's evaluate first correction term: $-3\lambda\beta V\ \bigcirc\!\bigcirc$ $\quad(\omega = \omega^2 = \vec p^2 + m^2$ ...$)$

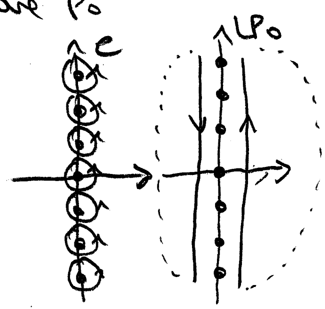
$$= -3\lambda\beta V\left(\frac{1}{\beta}\sum_n\int\frac{d^3p}{(2\pi)^3}D_0(\omega_n,\vec p)\right)^2 = -3\lambda\beta V\left(\frac{1}{\beta}\sum_n\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_n^2+\omega^2}\right)^2$$

Next neat trick for Matsubara summation: General task in complex frequency plane p_0

$$T\sum_{n=-\infty}^{\infty} f(p_0 = i\omega_n = 2\pi n T i)$$ rewrite as contour integral

that evaluates only at $p_0 = i\omega_n$. Need poles on imag. freq. axis. Use e.g. $\frac{1}{2}\beta\coth(\frac{1}{2}\beta p_0)$ analytic elsewhere

($\frac{1}{2}\beta\tanh(\frac{1}{2}\beta p_0)$ for fermions)



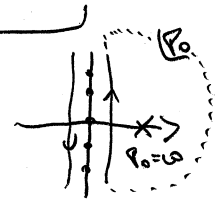
$$\Rightarrow \frac{T}{2\pi i}\oint_C dp_0\, f(p_0)\,\frac{1}{2}\beta\coth\left(\frac{1}{2}\beta p_0\right)$$ deform individ contours into two checked ones.

$$\frac{1}{2\pi i}\int_{i\infty-\varepsilon}^{-i\infty-\varepsilon} dp_0\, f(p_0)\left(-\frac{1}{2}-\frac{1}{e^{-\beta p_0}-1}\right) + \frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} dp_0\, f(p_0)\left(\frac{1}{2}+\frac{1}{e^{+\beta p_0}-1}\right)$$

combin ½ parts via $p_0\to -p_0$ in first int

f must NOT have singul. for imag. freq

$$T\sum_n f(p_0 = i\omega_n) = \underbrace{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp_0\,\frac{1}{2}\left[f(p_0)+f(-p_0)\right]}_{T=0} + \underbrace{\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} dp_0\left[f(p_0)+f(-p_0)\right]\frac{1}{e^{\beta p_0}-1}}_{T>0}$$



In our case $f(p_0) = \dfrac{-1}{p_0^2-\omega^2}$ $\quad$ (define $p_4 = ip_0$ $\quad d^4p = dp_4\,d^3p$)

$$T\sum_n\int\frac{d^3p}{(2\pi)^3}D_0(\omega_n,\vec p) = \int\frac{d^4p}{(2\pi)^4}\frac{1}{(p_4)^2+\vec p^2+m^2} + \int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega}\frac{1}{e^{\beta\omega}-1}$$

As we know from $T=0$ PT loop diagrams carry divergencies if naively eval. how does it behave? 4 dim sphere $p = \sqrt{p_4^2+\vec p^2}$ solid angle $\Omega_d = 2\pi^{d/2}[\Gamma(d/2)]^{-1}$

Since we are interested in taking derivatives, drop T independent term

$$\Rightarrow \sum_n \sum_p \ln[\ldots] = \sum_{\vec p} \int_1^{\beta\omega} d\lambda \left(\frac{1}{2} + \frac{\lambda}{e^{\lambda}-1}\right) \underset{\text{box}}{\overset{\text{large}}{\approx}} \frac{1}{2} V \int \frac{d^3p}{(2\pi)^3}\left[-\frac{1}{2}\beta\omega - \ln\left(1-e^{-\beta\omega}\right)\right]$$

Note: contains <u>zero-point energy</u> $E_0 = -\frac{\partial}{\partial\beta}\ln Z\big|_{T=0} = \frac{1}{2}V\int\frac{d^3p}{(2\pi)^3}\omega$ which needs to be <u>subtracted</u> when renormalizing e.g. pressure to physical $P|_{T=0}=0$

$P_0 = T\frac{\partial}{\partial V}\ln Z\big|_{T=0} = -E_0/V$.

$$\ln Z = \beta V(\mu^2-m^2)\xi^2 - \frac{1}{2}V\int\frac{d^3p}{(2\pi)^3}\left[\beta\omega + \underbrace{\ln\left(1-e^{-\beta(\omega-\mu)}\right) + \ln\left(1-e^{-\beta(\omega+\mu)}\right)}_{\text{particle \& antiparticle contrib.}}\right]$$

① Converges for $|\mu| \leq m$ ② θ drops out due to $U(1)$ sym. ③ ξ not yet fixed (related to prop. of zero mode) [For $m=0, \mu=0$ $P = \frac{\pi^2}{90}T^4$]

$s = 1+x+x^2$ $x\cdot s = x+x^2+x^3$
$s(1-x) = s - xs = 1$
$s = \frac{1}{1-x} = 1+x+x^2$ int.

$-\log(1-x) = x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots$

$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ $\zeta(4) = \frac{1}{90}$

Make <u>variational</u> ansatz at fixed β & μ $\frac{\partial\log Z}{\partial\xi} = 2\beta V(\mu^2-m^2)\xi \overset{!}{=} 0$

$\Rightarrow \xi = 0$ as long as $|\mu| < m$, undetermined at $\mu = m$.

What happens as $\mu \to m$? Check charge density $\sigma = \frac{T}{V}\frac{\partial\log Z}{\partial\mu}\Big|_{\mu=m} = \underbrace{2m\xi^2}_{\text{zero mode}} + \underbrace{\sigma^*(\beta,\mu=m)}_{\text{thermal contrib.}}$

here $\sigma^* = \int\frac{d^3p}{(2\pi)^3}\left(\frac{1}{e^{\beta(\omega-\mu)}-1} - \frac{1}{e^{\beta(\omega+\mu)}-1}\right)$ lower T at <u>fixed σ</u> $\mu \to m$

$\Rightarrow \xi^2 = \frac{\sigma - \sigma^*(\beta,\mu=m)}{2m}$ Phase transition with ξ^2 as order parameter (2nd order) and T_c defined implicitly $\sigma = \sigma^*(\beta_c, \mu=m)$ (Bose-Einstein condensation)

<u>What changes for fermions?</u> $T_\tau[\psi(\tau_1)\psi(\tau_2)] = \psi(\tau_1)\psi(\tau_2)\Theta(\tau_1-\tau_2) - \psi(\tau_2)\psi(\tau_1)\Theta(\tau_2-\tau_1)$

$\vec{S}(\tau+i\beta) = -\vec{S}(\tau)$ antiperiodicity in imag. time. Prohibits occurence of constant contribution in Fourier analysis, i.e. Matsubara freq. <u>$\omega_n = (2n+1)\pi T$</u>

$$Z = \prod_n \prod_{\vec p} \prod_{\alpha}^{\swarrow\text{Dirac index.}} \int i\, d\psi^+_{\alpha,n}(\vec p)\, d\psi_{\alpha,n}(\vec p)\, \exp\left[\sum_n\sum_p i\psi^+_{\alpha,n}(p)\, D_{\alpha\beta}\, \psi_{\beta,n}(p)\right]$$

with $D = -i\beta\left[(-i\omega_n+\mu) - \gamma^0\vec\gamma\cdot\vec p - m\gamma^0\right]$

To compute fermion Gaussian integrals briefly recall implementation via Grassmann numbers: $\{\eta,\eta\} = 0$ $\int d\eta = 0$ $\int d\eta\,\eta = 1$

Multidimensional Taylor expansion: $f = a + \sum_i a_i\eta_i + \sum_i b_i\eta_i^+ + \sum_{ij} a_{ij}\eta_i\eta_j + \ldots$

no terms with 2nd or higher power of the same variable!

$$\int d\eta_1^+ d\eta_1 \ldots d\eta_N^+ d\eta_N \exp\left[\vec\eta^+ D\vec\eta\right] = \det D \quad \text{use } \psi_\alpha(\vec x,\tau) = \frac{1}{\sqrt V}\sum_n\sum_p e^{i(\vec p\cdot\vec x+\omega_n\tau)}\tilde\psi_{\alpha n}(\vec p)$$

Now instead of 2×2 matrix for scalar we have 4×4 matrix D. Similarly

$$\ln Z = 2\sum_{n=0}^{\infty}\sum_p \ln\left[\beta^2\left[(\omega_n+i\mu)^2+\omega^2\right]\right] \quad \omega^2 = \vec p^{\,2}+m^2$$

$$= \sum_n\sum_p\left\{\ln\left[\beta^2(\omega_n^2+(\omega-\mu)^2)\right] + \ln\left[\beta^2(\omega_n^2+(\omega+\mu)^2)\right]\right\}$$

cutoff at $p<\Lambda$

$$\frac{1}{8\pi^2}\int_0^\Lambda \frac{dp\, p^3}{p^2+m^2} = \frac{1}{16\pi^2}\left[\Lambda^2 - m^2\ln\left(\frac{\Lambda^2+m^2}{m^2}\right)\right] \overset{\Lambda\gg m}{\approx} \frac{1}{16\pi^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right)\right]$$

need to renormalize! Since $T=0$ part additive to $T>0$ part subtract off to physical condition $\underline{P|_{T=0}=0}$.

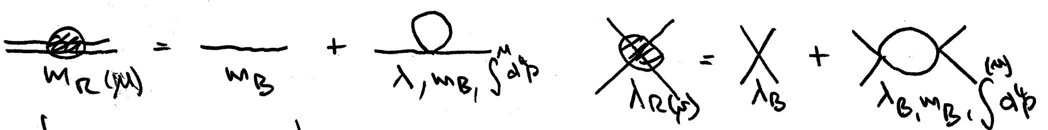
$$\ln Z_1 = -3\lambda\beta V\left(\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega}\frac{1}{e^{\beta\omega}-1}\right)^2 \Rightarrow P_1 = -3\lambda\left(\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega}\frac{1}{e^{\beta\omega}-1}\right)^2$$

For the pressure at $m=0$ $\quad P\approx P_0+P_1 = T^4\left(\frac{\pi^2}{90}-\frac{\lambda}{48}+\dots\right)$

But this expression as is not yet useful, since λ is bare parameter from within Lagrangian → unphysical quantity. Need to express via renormalized quantities.

<u>Brief excursion into renormalization</u>: Many physical systems are self-similar. The physics observed on different length scales can be described with the same d.o.f. but changes in parameters. In field theory experimentally measured masses and scattering rates define physical / renormalized parameters at a certain resolution scale. Theory can predict outcome at different resolution / energy scale

From T=0



μ energy scale of measurement.

For use in PT introduce so called counter terms which appear as new interaction terms:

wavefunction ren. $\quad$ coupl. renorm.

$$\mathcal{L} = \frac12\partial_\mu\varphi_B\partial_\mu\varphi_B - \frac12 m_B^2\varphi_B^2 - \lambda_B\varphi_B^4$$

what can change: $\varphi_B = Z_3^{1/2}\varphi_R$, $\lambda_B = Z_1 Z_3^{-2}\lambda_R$, $m_R^2 = m_B^2-\delta m^2$

rewrite

$$\mathcal{L} = \underbrace{\frac12\partial_\mu\varphi_R\partial_\mu\varphi_R - \frac12 m_R^2\varphi_R^2 - \lambda_R\varphi_R^4}_{\mathcal{L}_R} + \frac12\left[\partial_\mu\varphi_R\partial_\mu\varphi_R - m_R^2\varphi_R^2\right](Z_3-1) - Z_3\delta m^2\varphi_R^2 - \lambda_R\varphi_R^4(Z_1-1)$$

Need to <u>regularize</u> path integral (momentum cutoff, dimensional reg. or coord space reg. (lattice)). Then <u>renormalization</u> prescription provides

$$Z_1(\mu) = 1+\sum_n a_n\lambda_R^n \quad Z_3(\mu) = 1+\sum_m b_m\lambda_R^m \quad \delta m^2 = \sum_n c_n\lambda_R^n$$

Dim. reg. for $\ln Z_\Lambda$: work in $d = 4-\varepsilon$ dimensions, set $\varepsilon \to 0$ in end.

$$\lambda T \sum_n \int \frac{d^3p}{(2\pi)^3} D_0(\omega_n, \vec{p}) \rightsquigarrow \lambda \kappa^\varepsilon \int \frac{d^dq}{(2\pi)^d} \frac{i}{q^2 - m^2 + i\varepsilon}$$ here $[\kappa] = [m]$, κ^ε required to keep units consistent

$$= \frac{\lambda \kappa^\varepsilon}{(2\pi)^d} \frac{\pi^{d/2} \Gamma(1 - d/2)}{m^{2-d}}$$ use $\Gamma(-n+\delta) = \frac{(-1)^n}{n!}\left(\frac{1}{\delta} + \psi(n+1) + \mathcal{O}(\delta)\right)$ $\psi(n+1) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} + \gamma_E$ (Euler const.)

for small ε $\kappa^\varepsilon = \exp(\varepsilon \log \kappa) \approx 1 + \varepsilon \log(\kappa)$

$$\overset{\text{dim reg.}}{\rightsquigarrow} \frac{\lambda}{16\pi^2} m^2 \left[\frac{2}{\varepsilon} + \psi(2) + \ln\left(\frac{\kappa^2}{m^2}\right) + \ln(4\pi) + \mathcal{O}(\varepsilon)\right]$$

$$\text{cf. } \frac{\lambda}{16\pi^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right) + \mathcal{O}\left(\frac{m^4}{\Lambda^4}\right)\right]$$

in the pressure these T indep divergences removed by subtraction of $P|_{T=0}$.

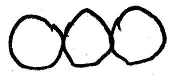
In the literature MS "minimal subtraction" scheme (on top of dim reg.) in two dues: counter term that includes only $1/\varepsilon$ term. In $\overline{MS}$ "bar" scheme also constant terms are absorbed.

Physical observables are independent of regularization prescription thus parameters can take on different values $\Lambda_{MOM}, \Lambda_{\overline{MS}}, \Lambda_{MS}$

To lowest order remember from $T=0$ $\lambda_B = \lambda_R + \mathcal{O}(\lambda_R^2)$

$$P = T^4\left(\frac{\pi^2}{90} - \frac{\lambda_R}{48} + \mathcal{O}(\lambda_R^2)\right)$$

Did we conquer the $T>0$ challenge to compute P in PT? Not yet!

From second order "ring" or "plasmon" diagrams 

Very peculiar: zero mode in central bubble set $n=0$ in massless theory $\propto \beta V (\int D_0)^2 T \int dp\, p^{-2}$ diverges for $p \to 0$. New divergence occurs at $T>0$ unrelated to UV and renormalization. (Discuss later)

Here just stating fact: field in a medium acquires "thermal mass" which upsets power counting. In scalar theory resum ring diagrams to obtain $$P = \frac{\pi^2}{90} T^4 \left[1 - \frac{15}{8}\left(\frac{\lambda}{\pi^2}\right) + \frac{15}{2}\left(\frac{\lambda}{\pi^2}\right)^{3/2} + \ldots\right]$$

power 3/2 unexpected! Note: since divergence arises from $n=0$ Bosonic mode, absent for fermions with $\omega_n = (2n+1)\pi T$.

what happens to the expansion? ($m=0$ case)

Basic building block of PT free propagator D_0. Question: is it a "good" anchor for expansion?

Look at full propagator $D^> = \beta^2 \langle \varphi_n(\vec{p})\, \varphi_{-n}(-\vec{p}) \rangle$ with $S = \sum \int \varphi_n(p)\, D_0^{-1} \varphi_n(p) + \dots$

$$= \beta^2 \int D\varphi\, e^{-S} \varphi_n(\vec{p})\, \varphi_{-n}(-\vec{p}) \Big/ \int D\varphi\, e^{-S} = -2 \frac{\delta \ln Z}{\delta D_0^{>-1}} = 2 D_0^{>2} \frac{\delta \ln Z}{\delta D_0}$$

Convenient to encode difference between $D_0^>$ and $D^>$ in self-energy Π

$$D^>(\omega_n, \vec{p}) = \left[\omega_n^2 + \vec{p}^{\,2} + m^2 + \Pi(\omega_n, \vec{p})\right]^{-1} = \left[1 + D_0^> \Pi\right]^{-1} D_0^>$$

since we looked at $\ln Z$, how is it related $\ln Z_0 = \frac{1}{2} \sum_n \sum_p \ln\left[D_0(\omega_n, \vec{p})\, \beta^{-2}\right]$

$$\Rightarrow \frac{\delta \ln Z_0}{\delta D_0} = \frac{1}{2} D_0^{>-1}(\omega_n, \vec{p})$$

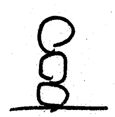
Full $D^> = 2 D_0^2 \frac{\delta \ln Z}{\delta D_0} = 2 D_0^2 \frac{\delta \ln Z_0}{\delta D_0} + 2 D_0^2 \frac{\delta \ln Z_I}{\delta D_0} = D_0 + 2 D_0 \frac{\delta \ln Z_I}{\delta D_0} D_0$

combine $\quad [1 + D_0 \Pi]^{-1} = 1 + 2 D_0 \frac{\delta}{\delta D_0}(3 \cdot \infty + \dots) \qquad \frac{\delta}{\delta D_0} \equiv$ "cut one line"

$= 1 + 12 D_0$  $+ O(\lambda^2)$

$\Pi_1 = 12$  $\quad \Pi_2 = -144$  $- 96$  only 1PI diagrams

$$\Pi_1 = 12 \lambda T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_n^2 + \omega^2} = 12\lambda \Big\{ \underbrace{\int \frac{d^4p}{(2\pi)^4} \frac{1}{p_4^2 + \vec{p}^{\,2} + m^2}}_{T^{vac}\text{ we do renormalization with mass counterterm}} + \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega} \frac{1}{e^{\beta\omega} - 1}}_{\text{for } m=0 \;=\; \lambda T^2 \text{ momentum independent "thermal mass"}} \Big\}$$

 $+$  $+$  $\sim \lambda\left(\frac{1}{k^2} + \frac{\lambda T^2}{k^4} + \frac{(\lambda T^2)^2}{k^6} + \dots\right) \rightsquigarrow \frac{1}{k^2 + \lambda T^2}$

actually ring diagrams give

$$\Pi_{ring}^{(n=0)} = -\frac{\lambda}{2} T \int \frac{d^3p}{(2\pi)^3} \frac{m_D^2}{(\vec{p}^{\,2} + m_D^2)^2} \simeq \underbrace{-\frac{\lambda^2}{4}\left(\frac{T^2}{12}\right)\left(\frac{T}{8\pi m_D}\right)}_{\sim \lambda^{3/2}}$$

Now QED: Perturbation theory requires gauge fixing to define D_0 photon Feynman propagator. How to do that in the path integral

Faddeev-Popov term $A_\mu \to A_\mu - \partial_\mu \alpha(\vec{x}, t)$ gauge fixing condition $F(A) = 0$ via functional delta func: $\int DA\, \delta(F) \det\left(\frac{\partial F}{\partial \alpha}\right) \exp[-S_E(A)]$

Neat trick rewrite Jacobian as new fields "ghosts" anti commuting scalars $\det\left(\frac{\partial F}{\partial \alpha}\right) = \int D\bar{c} \int Dc \exp\left[\int d\tau \int d^3x\, \bar{c} K^{-1} c\right]$

For photon gas: $\ln Z_0 = 4 \times (-\tfrac{1}{2}) \sum_n \sum_{\vec{p}} \ln[\beta^2(\omega_n^2 + \vec{p}^{\,2})] + 2 \times (+\tfrac{1}{2}) \sum_n \sum_{\vec{p}} \ln[\beta^2(\omega_n^2 + \vec{p}^{\,2})]$

the four d.o.f. inherent in A^μ reduced to physical two d.o.f. via ghost contribution.

First correction in presence of fermions (i.e. finite μ)

$\ln Z_2 = -\frac{1}{2}$ (photon line on electron loop e^-) $- \frac{1}{2}$ (photon line on ghost loop $c\bar{c}$) $\quad D_0^{\mu\nu} = \frac{1}{p^2}\left[g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right] \quad p_0 = 2\pi i n T$

$G_0 = \frac{1}{\not{p} - m} \quad p_0 = (2n+1)\pi T i + \mu$

Evaluation tedious but involves only known techniques. What changes at $T>0$? Gauge invariance unaffected $\Rightarrow$ Ward identity $k^\mu k^\nu D^{\mu\nu} = \xi$. BUT no more Lorentz invariance as heat bath or the medium singles out a rest frame. Decomposition of $D_{\mu\nu}$ affected: $g_{\mu\nu}$, k_μ and in addition $u_\mu = (1,0,0,0)$ indicates rest frame

$$D^{\mu\nu} = \underbrace{\frac{1}{G - k^2} P_T^{\mu\nu}}_{\text{magnetic}} + \underbrace{\frac{1}{F - k^2} P_L^{\mu\nu}}_{\text{electric}} + \frac{\xi}{k^2}\frac{k^\mu k^\nu}{k^2} \qquad \begin{array}{lll} @T=0 & F=G & G(k) = G(k^2) \\ @T>0 & F \neq G & G(k_0, \vec{k}) \end{array}$$

F, G denote poles & encode physical excitations of the system

$P_T^{ij} = \delta^{ij} - k^i k^j / \vec{k}^2 \qquad P_T^{00} = P_T^{0i} = P_T^{i0} = 0 \qquad P_L^{\mu\nu} = \frac{k^\mu k^\nu}{k^2} - g^{\mu\nu} - P_T^{\mu\nu}$

In QED: $G(n=0, \omega\to 0) = 0$ BUT $F(n=0, \omega \to 0) = e^2\left(\frac{T^2}{3} + \frac{\mu^2}{\pi^2}\right) = m_{el}^2$ magnetic modes remain massless (long ranged) while electric modes acquire a mass term proportional to eT. [No magnetic screening, this is why long wavelength approximation of QED: magneto-hydrodynamics (e.g. astrophysical applications)]

$$\frac{P}{T^4} = \frac{\pi^2}{45}\left(1 + \frac{7}{4} N_f\right) - \frac{5\pi^2}{72} \frac{\alpha(T) N_f}{\pi} + \mathcal{O}(\alpha^{3/2})$$

$$\alpha(T) = \frac{e^2(T)}{4\pi} = \frac{e^2}{4\pi}\left(1 + \frac{e^2 N_f}{6\pi^2} \ln\left(\frac{T}{\mu}\right)\right) + \mathcal{O}(e^6)$$

In QCD: force carrier gluons are self interacting. Gauge fields have additional color d.o.f. represented by $SU(N_c)$ group structure.

For the moment: $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu$ $\quad f_{abc}$ structure constants

From generators of $SU(N_c)$ $\quad [T^a, T^b] = i f^{abc} T^c \quad \mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta_{ab}$

$$\mathcal{L} = \sum_f \bar{\psi}^f_{\alpha,a} (i\not{\partial} - m - g\not{A}_a T^a)\psi^f_{\alpha,a} - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \qquad a \in 1\ldots N_c^2-1 \qquad T^a = \frac{\lambda^a}{2} \leftarrow \text{Gell-Mann matrices}$$

Switching off self interactions: ideal gas of N_c^2-1 gluons and N_f fermions.
$$P_0 = \frac{\pi^2}{45}(N_c^2-1)T^4 + \frac{N_c}{3\pi^2}\sum_f \int_0^\infty \frac{dp\, p^4}{E_p} n_F(p)$$

for $m_f = 0$ $\quad P_0(m_f=0) = N_c\left(\frac{7\pi^2 T^4}{180} + \frac{\mu_F T^2}{6} + \mu_F^4/12\pi^2\right)$

derivative coupling

First order corrections: $-\frac{1}{2}$  $-\frac{1}{2}$  $+\frac{1}{12}$  $+\frac{1}{8}$ 

Similar decomposition in longitudinal and transversal contributions
$$F^{PT}_{QCD}(n=0, \omega\to 0; m_f=0) = g^2\left[\left(\frac{1}{3}N_c + \frac{1}{6}N_f\right)T^2 + \frac{1}{2\pi}\sum_f \mu_f^2\right] = m_{el}^2$$

$G^{PT}_{QCD}(n=0, \omega\to 0, m_f=0) = 0$ $\quad$ BUT non-perturbatively $G_{QCD}(\omega\to 0) = m_m \neq 0$

$m^{QCD}_{el} \sim gT \qquad m^{QCD}_m \sim g^2 T \quad \Rightarrow$ upsets power counting in PT

When looking at Matsubara sums over propagators $f \sim \frac{1}{\omega_n^2 + E_p^2}$

$T\sum_n f \sim \frac{T}{E_p^2}$ enter with inverse powers of E_p. In PT expansion vertices enter with g^2 and T from mom. conservation.

$\Rightarrow$ Expansion parameter $\sim \frac{g^2 T}{E_b}$ largest possible $\frac{g^2 T}{m} \leftarrow$ smallest mass

In QED $\quad m_{small} \sim eT \Rightarrow \frac{e^2 T}{eT} \sim e$ small

QCD $\quad m_{small} \sim g^2 T \Rightarrow \frac{g^2 T}{g^2 T} \sim 1$ large

generation of thermal magnetic mass invalidates PT $\Rightarrow$ "Linde problem" of QCD: Remedies:

- $\Rightarrow$ clever resummations: Hard-Thermal-loop PT (e.g. N. Su arXiv:1204.0260)
- $\Rightarrow$ non-pert. effective theories for zero modes: dim. reduced EQCD (e.g. Prog. Part Nucl. Ph. 67 (2012) 168) J. Roller
- $\Rightarrow$ fully non-perturbative simulations: lattice QCD

Lattice QCD {Ref: Gattringer & Lang, Rothe, Smit, Montvay & Münster}

Spacetime regularization of QCD on 4-dim hypercube. Discrete $(\vec{x},\tau) = (\vec{n}_s a_s, n_\tau a_\tau)$ $n_s^i, n_\tau \in \mathbb{N}$ a_s, a_τ spatial, temporal lattice spacing

Central concept: gauge invariance $\equiv$ free to choose basis for the colour space of quarks at each $(\vec{x},\tau)$. Role of gauge field: book keeping device keeps track of colour coordinate systems.

$\psi_\alpha^A(x)$ α Dirac index A fundamental SU(3) rep. index $\{1,2,3\}$ $\psi = \begin{pmatrix} r \\ g \\ b \end{pmatrix}_\alpha (\vec{x},\tau)$

Color rotations via multiplication with SU(3) group element.

Constructing gauge invariant action for quarks $\to$ necessitates gauge field

$\psi_x' \overset{GT}{=} \Omega_x \psi_x$ naively $\Delta_\mu \psi_x = (\psi_{x+a\hat{\mu}} - \psi_x)/a$ not homogeneous

i.e. $(\Delta_\mu \psi_x)' \neq \Omega_x (\Delta_\mu \psi_x)$ Instead need correction terms

$$D_\mu \psi_x = \frac{1}{a}(\psi_{x+a\hat{\mu}} - \psi_x) - i(\tilde{C}_{x,\mu}\psi_x + C_{x,\mu}\psi_{x+a\hat{\mu}})$$ choose $\tilde{C}, C$ such that $(D_\mu \psi_x)' = \Omega_x (D_\mu \psi_x)$

$$\Omega_x\left[\frac{1}{a}(\psi_{x+a\mu} - \psi_x) - i(\tilde{C}_{x,\mu}\psi_x + C_{x,\mu}\psi_{x+a\hat{\mu}})\right.$$
$$\doteq \Omega_x\left[\frac{1}{a}(\Omega^+_{x+a\mu}\psi'_{x+a\mu} - \Omega^+_x \psi_x) - i(\tilde{C}_{x,\mu}\Omega^+_x\psi'_x + C_{x,\mu}\Omega^+_{x+a\hat{\mu}}\psi'_{x+a\hat{\mu}})\right]$$

No correction needed for Ω_x: $\tilde{C}_{x,\mu} = 0$ But:

$$C'_{x,\mu} = \Omega_x C_{x,\mu}\Omega^+_{x+a\mu} + \frac{i}{a}(\Omega_x \Omega^+_{x+a\mu} - 1) = \Omega_x C_{x,\mu}\Omega^+_{x+a\mu} + \Omega_x i \Delta_\mu \Omega^+_x$$

transformation looks similar to continuum gauge pot.

Can we build a field strength tensor from it? $F_{\mu\nu} = [D_\mu, D_\nu]$ (cont.)

$$C_{x,\mu\nu} = D_\mu C_{x,\nu} - D_\nu C_{x,\mu} = \frac{1}{a}(C_{x+a\mu,\nu} - C_{x,\nu}) - i C_{x,\mu}C_{x+a\mu,\nu} - \left(\begin{smallmatrix}\text{same with}\\ \text{replacement}\\ \mu\leftrightarrow\nu\end{smallmatrix}\right)$$

How does it transform: $C'_{x,\mu\nu} = \Omega_x C_{x,\mu\nu}\Omega^+_{x+a\hat{\mu}+a\hat{\nu}}$ GREAT!

can combine two into a gauge invariant expression $\mathrm{Tr}[C_{x,\mu\nu} C^+_{x,\mu\nu}]$.

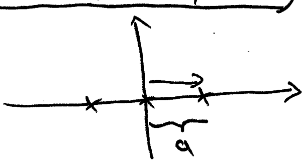
How to parametrize $C_{x,\mu}$? Take a look at pure gauge contribution:

$C_{x,\mu} = 0 \Rightarrow C'_{x,\mu} = \frac{i}{a}(\underbrace{\Omega_x \Omega^+_{x+a\mu}}_{SU(3)} - 1) \Rightarrow C_{x,\mu} = \frac{i}{a}(U_{x,\mu} - 1)$ then

we get for new U quantity $U'_{x,\mu} = \Omega_x U_{\mu,x}\Omega^+_{x+a\mu}$ "connects gauge transformations at neighbouring sites". Since SU(3) matrix can be written as $U_{x,\mu} = \exp[-i\kappa\, G^a_{x,\mu} T^a]$ T^a Gell-Mann $T = \lambda/2$

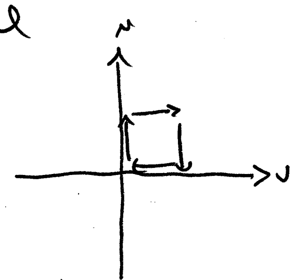
Role of $U_{x,\mu}$ clear when plugging back into $\underline{D_\mu \psi_x = \frac{1}{a}\left(U_{x,\mu}\psi_{x+a\hat\mu} - \psi_x\right)}$

$U_{x,\mu}$ implements <u>parallel transport</u> of color space between x and $x+a\hat\mu$. In continuum such parallel transport is written as $U(x,y) = P\exp\left[-ig\int_{C_{x,y}} dz_\mu A_\mu(z)\right]$ "Wilson line"



for small lattice spacing approximate via midpoint rule

$$U_{x,\mu} = \exp\left[-iag\, A^a_\mu(x+\tfrac{a}{2}\hat\mu)\, T^a\right]. \qquad (\sqcap + \sqcup)(\sqcap + \sqcup)^\dagger$$



With this parametrization: action $\mathrm{Tr}\left[C_{x,\mu\nu} C^\dagger_{x,\mu\nu}\right]$

$$= \frac{1}{a^4}\mathrm{Tr}\left[2 - P_{x,\mu\nu} - P^\dagger_{x,\mu\nu}\right] \qquad P_{x,\mu\nu} = U_{x,\mu}\, U_{x+a\hat\mu,\nu}\, U^\dagger_{x+a\nu,\mu}\, U^\dagger_{x,\nu} \quad \underline{\text{plaquette}}$$

$$S[U] = \underbrace{\frac{2N_c}{g^2}}_{\beta_L} \sum_{x,\mu,\nu} \frac{1}{N_c}\mathrm{Re}\,\mathrm{Tr}\left[1 - P_{x,\mu\nu}\right]$$

isotropic Wilson action (1 param.)

$$S_a[U] = \frac{1}{N_c}\sum_x \left\{ \frac{2}{\beta_\tau}\frac{a_s}{a_\tau}\sum_i \mathrm{Re}\,\mathrm{Tr}\left[1 - P_{x,0i}\right] + \frac{2}{\beta_s}\frac{a_\tau}{a_s}\sum_{i<j}\mathrm{Re}\,\mathrm{Tr}\left(1 - P_{x,ij}\right)\right\}$$

anisotropic Wilson action (2 indep. param.)

Need to construct a path integral over the SU(3) group. Two possibilities: directly in group space, in generator space.

$$Z = \int DU\, e^{-S[U]} \quad \text{with} \quad \int DU = \int \prod_{x,\mu} dU_{x,\mu}$$

Note: SU(3) is a compact group with a finite volume, so for finite $a_{s,\tau}$ and $N_{s,\tau}$ $\Rightarrow$ Z well defined.

Need to find a <u>gauge invariant measure</u>: Haar measure. $\int DU = 1$

$$dU = \underset{\substack{\text{color}\\ \text{space metric}}}{N \cdot \sqrt{\det g}}\; \prod_k \underset{\text{coordinates}}{d\alpha_k} \qquad \underset{(1,\dots,8)}{g_{kl}} = \mathrm{Tr}\left[\frac{\partial U}{\partial \alpha^k}\frac{\partial U^\dagger}{\partial \alpha^l}\right]$$

<u>Beware</u> $\frac{d}{d\alpha}e^{\alpha M} = Me^{\alpha M}$ BUT $\frac{d}{d\alpha}e^{\alpha M + \beta N} \neq Me^{\alpha M+\beta N}$

Gauge transformations $\equiv$ <u>translations</u> in group space $\tilde U \overset{l}{=} \Omega U$ $\;\bar U \overset{r}{=} U\Omega$

$\alpha^k \to f^k(\alpha)$ $\quad g_{kl} = \underbrace{g_{mn}\frac{\partial \alpha'^m}{\partial \alpha^k}\frac{\partial \alpha'^n}{\partial \alpha^l}}$, cancels with Jacobian det. change of vars.

$\Rightarrow d(\Omega U) = d(U\Omega) = dU$. If written explicitly in $A^a_\mu(x)$:

define $(\bar A_\mu)_{bc}(x) = a \cdot g\, f_{bcd}\, A^d_\mu(x)$ $\qquad g_{kl,\mu} = \left[\frac{2}{\bar A_\mu}\sinh\left(\frac{\bar A_\mu}{2}\right)\right]^2_{kl}$

How to evaluate this path integral non-perturbatively?

$\Rightarrow$ MONTE CARLO METHODS. [only glimmer def. due to insufficient time.]

General problem: compute <u>highly dimensional integral</u>:

$\int Dx = \int \prod_i dx_i$ often as $\int Dx\, f(x)\, P(x) / \int Dx\, P(x) = \int Dx\, f(x) \frac{P(x)}{\int Dx\, P(x)}$ } positive & normalized

prototype: $I = \int_0^1 dx_1 \ldots \int_0^1 dx_d\, g(x_1, \ldots x_d) " = " \int_0^1 Dx\, g(x)$

<u>Naive integration</u>: divide axes in equidistant grids $\Rightarrow$ # of points N^d

errors: <u>Midpoint</u>: $(b-a) f(\frac{a+b}{2})$ $\delta I \sim N^{-1}$ <u>Trapezoid</u>: $\frac{1}{2}(b-a)[f(b)+f(a)]$ $\delta I \sim N^{-2}$

<u>Simpson</u>: average of prev. two $\delta I \sim N^{-4}$

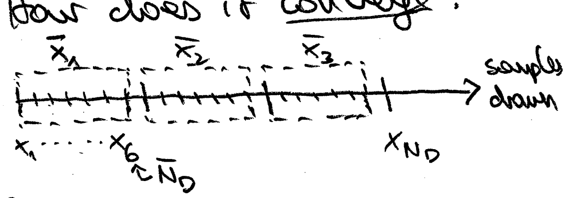
sample label $1 \ldots N_D$

<u>Naive Monte-Carlo</u>: generate N_D vectors $(x_1, \ldots, x_d) = \tilde{x}_k$ with $x_i \in \mathrm{Unif}(0,1)$

<u>uniform random numbers</u>. Estimate $I_{N_D} = \frac{1}{N_D} \sum_{k=1}^{N_D} g(\tilde{x}_k)$ due to law

of large numbers $\lim_{N_D \to \infty} \frac{1}{N_D} \sum_{k=1}^{N_D} (x_i)_k = \int dx_i\, x_i\, P(x) = \langle x_i \rangle$ thus we have

$\lim_{N_D \to \infty} I_{N_D} = \prod_i \int dx_i\, g(x_1, \ldots, x_d)\, P(x_1) \ldots P(x_d) = I$ How does it <u>converge</u>?



Remember: $I_{\bar{N}_D}$ becomes a random variable itself

Use central limit theorem: if variance of x_k's finite then taking $\bar{N}_D$ large makes $\bar{x}_k$ Gaussian distributed:

$\sqrt{\tilde{N}_D}\,(\bar{x} - \langle x \rangle) \rightsquigarrow \mathrm{Normal}(0, \langle x^2 \rangle)$ $\tilde{N}_D = N_D / \bar{N}_D$

$$\mathrm{Var}(I_{N_D}) = \mathrm{Var}\left(\frac{1}{\tilde{N}_D} \sum_k g(\bar{x}_k)\right) \overset{\text{indep samples}}{=} \frac{1}{\tilde{N}_D^2} \sum_{k=1}^{\tilde{N}_D} \mathrm{Var}(g(\bar{x}_k)) \overset{\text{const var.}}{=} \frac{1}{\tilde{N}_D^2} \tilde{N}_D \sigma^2 = \frac{\sigma^2}{\tilde{N}_D}$$

$\delta I \propto \frac{1}{\sqrt{N_D}}$ INDEPENDENT OF DIMENSION!

Compare to naive int: $N^d = N_D$ (for same # of function evaluations)

$N = (N_D)^{1/d}$ vs. Simpson $\frac{1}{\sqrt{N_D}} < \frac{1}{N^4} = \frac{1}{N_D^{4/d}}$ $\Rightarrow$ MC wins for $d > 8$!

Further improvement: reduce variance of g, i.e. σ. "Sample preferably around points $(x_1, \ldots, x_d)$ that contribute significantly to integral."

$$I = \int_a^b dx\, f(x) = \int_a^b \frac{f(x)}{\omega(x)} \underbrace{\omega(x)\, dx}_{\text{Normalized}} \longrightarrow \text{find } y \text{ with } \frac{dy}{dx} = \omega(x) \longrightarrow \int_{y(a)}^{y(b)} \frac{f(y)}{\omega(y)} dy \approx \frac{1}{N_D} \sum_{y_i} \frac{f(y_i)}{\omega(y_i)} = I_{N_D}^{(y)}$$

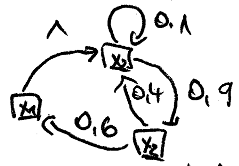
The closer $\omega(x)$ is to $f(x)$ the smaller the variance of $I_{N_D}(y)$.

In practice $\int dx\, f(x) \left(e^{-S} / \int dx\, e^{-S}\right)$ need y w/ $\frac{dy}{dx} = \frac{e^{-S}}{\int dx\, e^{-S}} \rightarrow \int dy\, f(y)$

<u>Markov-Chain Monte Carlo</u>: Generate a random process whose realizations are <u>distributed</u> according to a target distribution.

$x_0 \xrightarrow{\text{MC update}} x_1 \longrightarrow x_2 \longrightarrow$ MC time

For a large number of draws $P(x) \approx \Pi(x)$. Here, next draw depends (if at all) only on previous draw. Analyzed via graphs



Two important ingredients: ① irreducibility: for any state x_k $\exists$ finite prob. to visit all other states ② aperiodicity: takes finite # of steps to return to any state ①+② $\Rightarrow$ ERGODIC samples faithfully the whole state space. Prerequisite to converge to a stationary distribution $\Pi(x)$: $P_0(x) \to P_1(x) \to \ldots \to \Pi(x)$ $\leftarrow$ indep. of $P_0(x)$.

How to construct such a process converging to pre-defined $\Pi(x)$?

Detailed Balance: $\Pi(x)\, P(x'|x) = \Pi(x')\, P(x|x')$

(can show that $\Pi(x)$ is Eigenvector to transition operator w/ Eval 1)

Ground-breaking implementation: Metropolis-Hastings algorithm

① Propose new d.o.f. x' (proposal step). ② Accept or reject new configuration based on $P_{AR} = \min\left\{1, \frac{P(x')}{P(x)}\right\}$ $\Rightarrow$ guaranteed to approximate $P(x)$.

Note: - Since only $P(x')/P(x)$ needed, we do not need to know the normalization of P. If $P(x) = e^{-S}$ then normalization is Z.

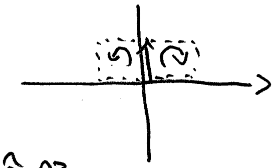
- Evaluating observables: $\langle O(U)\rangle = \lim_{N_D\to\infty} \frac{1}{N_D}\sum_k O(U_k) = \int DU\, O(U)\, \frac{e^{-S}}{\int DU e^{-S}}$ cannot evaluate Z itself.

- Proposals can be efficiently generated via simple distributions $\Rightarrow$ works well even in high-dim settings.

- Since started often from non-representative x_0 (hot/cold start) need to allow "burn-in" time before using x_k's for estimation of $\langle O\rangle$. Beware subsequent samples may be highly correlated. [~~Trade~~ tradeoff with acceptance rate]

Concrete example $T_{AR} = \min\{1, \exp[S(x) - S(x')]\}$ $P(x) \propto e^{-S}$. If S decreases, accept proposal with 100%; if it increases accept with prob. $\exp(\Delta S) < 1$. (Draw uniform random number $\in [0,1]$, if larger than $e^{\Delta S}$ accept). If action is local, updating only single d.o.f often most efficient, simple expr. for $S(x) - S(x')$.

For Wilson action, change of $U_\mu(x)$ affects all six connected plaquettes: e.g. $U_{x,0}$ – $P_{x\,01}, P_{x,02}, P_{x,03}$
$P^+_{x-a\hat{1},01}, P^+_{x-a\hat{2},02}, P^+_{x-a\hat{3},03}$



loop over x, y, z, τ, $\mu = 0, \dots, 3$

at each step propose a new SU(3) matrix [efficient: multiply current matrix with random matrix close to unity). Compute affected plaquette sums $\Rightarrow$ accept–reject.

Brief Interlude: Scale setting Path integral on lattice contains all quantum fluct "= loops" that fit into the box and are resolved on the grid. Output of simulation is dimensionless number $\to$ how to connect to physical quantities? If close enough to continuum-limit compare physical quantities such as masses and couplings to lattice counterpart. Involves assigning physical units to lattice spacing. {For comprehensive overview: R. Sommer arXiv: 1401.3270}

At T=0: $\langle \phi(\tau)\phi(0)\rangle = \sum_m \langle 0|e^{\tau H}\phi(0)\,e^{-\tau H}|m\rangle\langle m|\phi(0)|0\rangle = \sum_m e^{-\tau(E_m - E_0)}\,|\langle 0|\phi|m\rangle|^2$

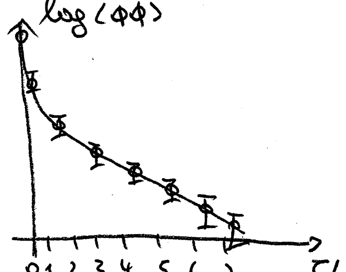
correlation functions = sum of exponentials. At large τ the smallest relevant $(E_m - E_0)$ survives. For $|\langle 0|\phi|m\rangle|^2$ to give non-zero result $|m\rangle$ needs to contain at least one particle $\Rightarrow (E_1 - E_0)$ survives.

Projected to $\vec{p}^2 = 0$ $\quad (E_1 - E_0)_{\vec{p}=0} = m_{\text{ground state}} \leftarrow$ the pole mass which includes all loop corrections.

How to compare to "physical mass" from PDG?

$$\exp(-m^{\text{phys.}}\,\tau^{\text{phys}}) = \exp\Big(-\underbrace{(m^{\text{phys}}a)}_{m^{\text{num}}}\underbrace{(\tau^{\text{phys}}/a)}_{\in \mathbb{N}}\Big)$$

$$m^{\text{num}} = m^{\text{phys}}\,a \quad \Rightarrow \quad a = \frac{m^{\text{num}}}{m^{\text{phys}}} \qquad (0{,}197 \text{ fm GeV} = 1)$$



Now ready to compute pressure on the lattice:

① choose observable related to pressure that is easy to simulate
② estimate it to adequate statistical precision
③ subtract T=0 contribution to renormalize (compatible with $P_{T=0}=0$)
④ compute pressure from renormalized results
⑤ repeat at different lattice spacings $\to$ continuum limit extrapolation
{Ref. Engels et. al Nucl. Phys B. 205 545 (1982); Ejiri et. al. PRD 79 051501 (2009)}

Challenge: cannot compute pressure directly $P = \frac{T}{V}\log(Z)$ since MC normalized by Z. Instead use derivatives of Z, which correspond to observables that can be simulated.

$$\varepsilon = \frac{1}{V}\frac{\partial}{\partial\beta}\log(Z)\Big|_{V=\text{const}} \qquad P = -\frac{1}{T}\frac{\partial}{\partial V}\log(Z)\Big|_{T=\text{const}}$$

turns out useful quantity is trace anomaly: $\frac{\varepsilon-3P}{T^4} = T\frac{\partial}{\partial T}\left(\frac{P}{T^4}\right)$ so that $\frac{P}{T^4} = \int_{T_0}^{T} dT\,\frac{\varepsilon-3P}{T^4}$

$(T^\mu_\mu = \varepsilon - 3P)$

choosing $P(T_0)=0$. Derivatives w.r.t. physical spatial box size (V) and imaginary time extent ($1/T$) as derivatives w.r.t. physical lattice spacing.

Note: In anisotropic Wilson action $\exists$ 2 indep. bare parameters. Convention is to set a_s^B and a_t^B to unity and encode difference between temporal & spatial direction via couplings g_s^B and g_t^B.

$$Z[\overbrace{V,T}^{\text{phys.}}] = Z[a_s^P, a_t^P] = Z\left[a_s^P, \xi^P = \frac{a_s^P}{a_t^P}\right] = Z\left(a_s^P(g_s^B, g_t^B), \xi^P(g_s^B, g_t^B)\right)$$

$$\frac{\partial}{\partial\beta^{\text{phys}}} = \frac{1}{N_t}\frac{\partial}{\partial a_t^{\text{phys}}}\Big|_{a_s^P=\text{const}} = -\frac{1}{N_t}(\xi^P)^2\frac{1}{a_s^P}\frac{\partial}{\partial\xi^P}\Big|_{a_s^P=\text{const}}$$

$$\Rightarrow \varepsilon = -\frac{1}{N_x^3(a_s^P)^3}\frac{1}{N_t}(\xi^P)^2\frac{1}{a_s^P}\frac{\partial\log Z}{\partial\xi^P}\Big|_{a_s^P=\text{const}}$$

$$P = -\frac{1}{N_t(a_t^P)}\frac{1}{3}\frac{1}{N_x^3(a_s^P)^3}\left\{\frac{\partial\xi^P}{\partial a_s^P}\frac{\partial\log Z}{\partial\xi^P}\Big|_{a_s=\text{const.}} + \frac{\partial\log Z}{\partial a_s^P}\Big|_{\xi^P=\text{const}}\right\}$$

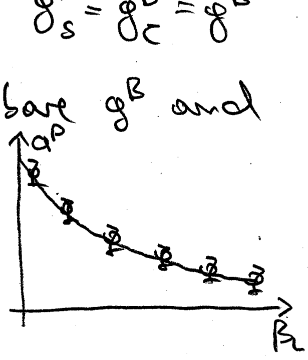
combining into trace anomaly cancels two contributions

$$\varepsilon - 3p = -\frac{\xi^P}{N_t N_x^3}\frac{1}{(a_s^P)^3}\frac{\partial\log Z}{\partial a_s^P}\Big|_{\xi^P=\text{const}}$$

we choose simplest scenario $\xi^P = 1 \Leftrightarrow \xi^B = 1 \Leftrightarrow g_s^B = g_t^B = g^B$

Now we need a trick to evaluate $\frac{\partial\log Z}{\partial a_s^P}$. In practice: select bare g^B and via scale setting find corresponding $a_s^P = a_t^P = a^P(\beta_L)$

Invert this relation numerically to obtain $\frac{\partial\beta_L}{\partial a^P}$



"lattice β-function" $\frac{\partial}{\partial a_s^P} = \frac{\partial\beta_L}{\partial a_s^P}\frac{\partial}{\partial\beta_L}$

$$\frac{\partial\log Z}{\partial a_s^P} = \frac{\partial\beta_L}{\partial a_s^P}\frac{1}{Z}\frac{\partial Z}{\partial\beta_L} = \frac{\partial\beta_L}{\partial a_s^P}\frac{1}{Z}\int DU\,\exp\left[-\beta_L\sum_{x,\mu,\nu}(1-P_{x,\mu\nu})\right]\left\{\sum_{x,\mu,\nu}(1-P_{x,\mu\nu})\right\}$$

on the lattice often compute normalized spatial or temporal plaquette avg.

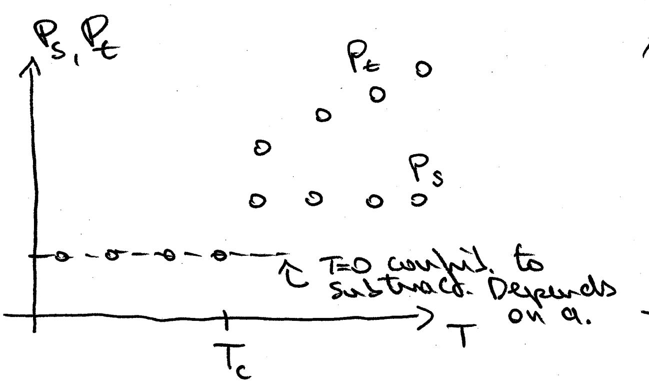
$$P_\tau = \sum_x \sum_i \mathrm{Re\,Tr}\left[P_{x,0i}\right]/(N_c\cdot U\cdot T\cdot N_{dim})$$
$$\gamma = 3$$
$$P_s = \sum_x \sum_{i<j} \mathrm{Re\,Tr}\left[P_{x,ij}\right]/(N_c\cdot U\cdot T\cdot N_{dim})$$

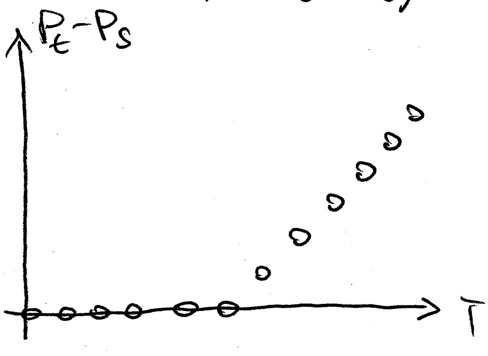
normalized that $P_\tau = P_s = 1$ if $U = 1$.

$$\frac{\varepsilon - 3P}{T^4} = N_\tau^4 \left(a^\rho \frac{\partial \beta_L}{\partial a^\rho}\right) N_{dim} N_c \left[(1-P_t) + (1-P_s) \right]$$
$$= N_\tau^4 \left(a^\rho \frac{\partial g^{B-2}}{\partial a^\rho}\right) 2 N_d N_c \left[(1-P_t) + (1-P_s) \right]$$

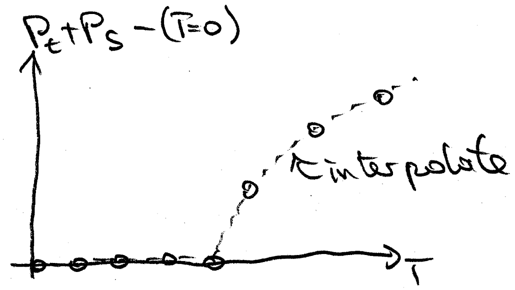
Need to subtract the vacuum contributions to get finite result in the continuum limit, i.e. renormalize correctly! Happens on the level of free energies: $F_{ren} = F - F_{T=0} = \log(Z/Z_0)$ same as setting $P_{ren} = P - P_{T=0}$. Thus we need to subtract off the T=0 (or at least very low T contribution)

$$\Rightarrow \left.\frac{\varepsilon - 3P}{T^4}\right|_{ren} = N_\tau^4 \left(a^\rho \frac{\partial (g^{B})^{-2}}{\partial a^\rho}\right) 2 N_d N_c \left[\left(P_t^{T=0} - P_t\right) + \left(P_s^{T=0} - P_s\right) \right]$$

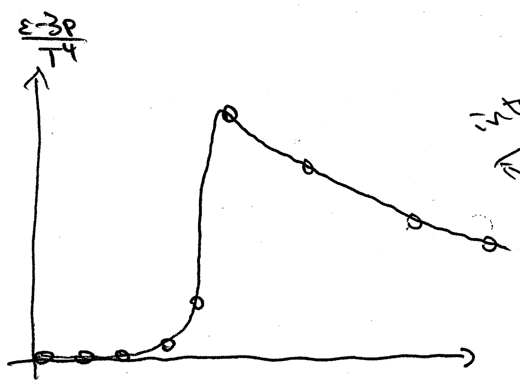




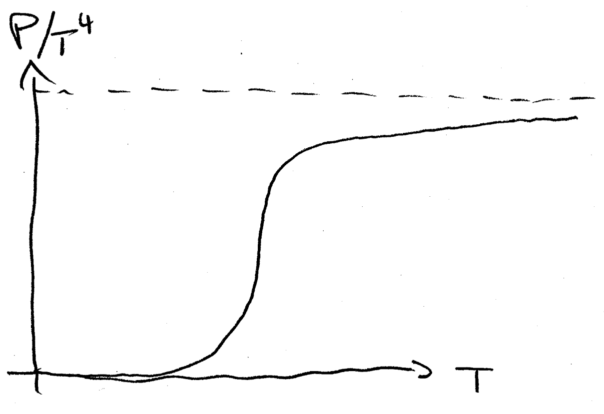
nice to identify phase transition. (c.f. Polyakov loop as order parameter of SU(N))



purely gluonic theory has 1st order phase transition at $T_c = 271$ MeV
full QCD crossover transition at $T_{co} = 155$ MeV



integrate
interpolation
→



Stephan-Boltzmann limit